Issn:0972-2440

THE MATHEMATICS TEACHER

YEAR - 2014

VOLUME - 50
ISSUES 1&2

THE ASSOCIATION OF MATHEMATICS TEACHERS OF INDIA

The Association of Mathematics Teachers of India (AMTI) was started in 1965 for the promotion of efforts to improve Mathematics education at all levels. A major aim of the Association is to assist practising teachers of Mathematics in schools in improving their expertise and professional skills. Another important aim is to spot out and foster Mathematical talents in the young. The Association also seeks to disseminate new trends in Mathematics education among parents and public. Other activities of the Association include consultancy services to schools in equipping the Mathematics section of their libraries, in organising children's Mathematics clubs and fairs, in setting up teacher centres in schools, in conducting Mathematics laboratory programmes, in holding practical tests in Mathematics in assisting children in participating investigational projects.

The Association holds " National Mathematical Talent Search Competition " annually and organizes Orientation Courses, Seminars and Workshops for teachers and courses for talented students. A national conference is held annually in different parts of the country for teachers to meet and deliberate on important issues of Mathematics education. Innovative teacher award has been instituted to give public recognition to enterprising and pioneering teachers of Mathematics for which entries from teachers are invited.

An award for contributions to the Mathematics Teacher relating to History of Mathematics in the context of mathematics education had been instituted by Prof.R.C.Gupta.

"The Mathematics Teacher (India)" (MT) is the official quarterly journal of the Association and is issued twice a year. It has been approved for use in schools and colleges of education by the Government departments of education in many States. Besides MT the Association also brings out Junior Mathematician (JM), three issues in a year, especially for school students in English and Tamil.

The membership of the Association is open to all those interested in Mathematics and Mathematics Education. The membership fee inclusive of subscription for "**The Mathematics Teacher (India)**" and effective from April 1993 is as follows:

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Life	Rs. 500	Rs. 1000
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The Journal "The Mathematics Teacher" will be supplied free to all members.
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The Mathematics Teacher
(INDIA)

OFFICIAL JOURNAL OF THE ASSOCIATION OF MATHEMATICS TEACHERS OF INDIA

Volume: 50 Issues: 1 & 2 Year: 2014

Dr. S.Muralidharan
Editor

EDITORIAL

This issue contains problems and solutions for the National Mathematics Talent Competitions conducted by the Association of Mathematics Teachers of India as well as solutions to the Regional and National Olympiads, 2013. The solutions were contributed by S.R.Santhanam and M.Mahadevan. Some of the solutions were modified by the Editor who also provided alternate solutions wherever feasible.

It is hoped that the students and teachers will find this issue useful. Any suggestions for improvement are welcome.

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SCREENING TEST – GAUSS CONTEST
NMTC at PRIMARY LEVEL
V & VI Standards

PART - A

1. Consider the numbers 2,3,4,5. Form two digit numbers of different digits using these numbers. How many of them are odd?

(A) 4 (B) 5 (C) 6 (D) 7

Solution: Possible numbers are

$$\underline{23},\ 24,\ \underline{25},\ 34,\ \underline{35},\ \underline{45},\ 32,\ 42,\ 52,\ \underline{43},\ \underline{53},\ 54$$

6 of them are odd as indicated.

One can also argue as follows: If the number is odd, then its unit digit is necessarily odd. Hence we can choose only 3 or 5 for the units digit. Once the units digit is chosen, the digit can be any one of the three remaining numbers. Hence there are 3×2 numbers that are odd.

Answer: (C)

2. A is the sum of all even three digit numbers in which all the three digits are equal. B is the sum of all odd three digit numbers in which all the digits are equal. The value of $\frac{B}{A}$ is

(A) $\frac{5}{4}$ (B) $\frac{4}{3}$ (C) $\frac{6}{5}$ (D) $\frac{7}{6}$

Solution: The possible three digit even numbers with equal digits are 222, 444, 666, 888 and their sum is $A = 2220$.

The possible three digit odd numbers with equal digits are 111, 333, 555, 777, 999 with sum $B = 2775$. Hence

$$\frac{B}{A} = \frac{2775}{2220} = \frac{925}{740} = \frac{185}{148} = \frac{5}{4}$$

Answer: (A)

3. 21 rose plants, 42 sunflower plants and 56 dalia plants have to be planted in rows such that each row contains the same number of plants ofone variety only. The minimum number of rows in which the above plants may be planted is

   (A) 3 (B) 15 (C) 17 (D) 21

   Solution: If there are d rows, then d must divide 21, 42 and 56. The least number of rows will be required if we plant n plants of each variety where n is the largest integer that divides 21, 42 and 56. Since the largest n is 7, we need a minimum of 3 rows.

   Answer: (A)

4. The length and breadth of a square are increased by 30% and 20% respectively. The area of the rectangle so formed exceeds the area of the square by

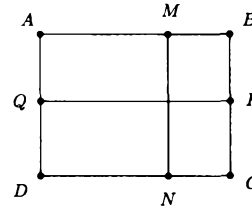
   (A) 25% (B) 50% (C) 60% (D) 56%

   Solution: Let a be the side of the square. The increased length is $\frac{130a}{100}$ and increased breadth is $\frac{120a}{100}$.

   Area of increased figure is $\frac{130 \times 120}{10000}a^2 = \frac{156}{100}a^2$. Hence the area exceeds the area of the square by 56%

   Answer: (D)

5. $ABCD$ is a rectangular play ground in which $AB = 40\,\text{m}$ and $BC = 30\,\text{m}$. The Physical director of the school gave punishment to two students Samrud and Saket. Samrud has to start from A and go round the play ground along $ABCDA$ twice.



Saket has to start from A go along AM, MN, NC, CP, PQ and to A two times. MN is perpendicular to CD and PQ perpendicular to DA. Then

(A) Samrud covers more distance than Saket

(B) Saket covers more distance than Samrud

(C) Samrud and Saket cover equal distances

(D) Saket covers exactly twice the distance Samrud covers.

Solution: Distance covered by Samrud is

$$(40 + 30 + 40 + 30) \times 2 = 280 \text{ m}$$

Distance covered by Saket is

$$\begin{aligned} &2 \times (AM + MN + NC + CP + PQ + QA) \\ &\quad = 2 \times ((AM + NC) + (CP + QA) + MN + PQ) \\ &\quad\quad = 2 \times (AB + AD + AD + AB) = 280 \text{ m} \end{aligned}$$

Hence they cover equal distances.

Answer: (C)

6. B has 5 Rupees more than C. A has 14 Rupees more than B. Which transaction makes equal money to all the three.

(A) A gives 6 Rs. to B and B gives 3 Rs. to C

(B) A gives 3 Rs. to B and C receives 6 Rs. from A

(C) A gives 2 Rs. to C and B gives 5 Rs. to C

(D) A gives 8 Rs. to C and B receives 3 Rs. from A

Solution: Let C have Rs. x. Then B has Rs. $5 + x$ and A has Rs. $19 + x$. The total money they have is $x + 5 + x + 19 + x = 24 + 3x$. Hence if they need to end up with equal amounts, then each will have Rs. $8 + x$. Clearly only transactions in (D) makes all to have equal money.

Answer: (D)

7. A natural number a is multiplied by 11 and 33 is added to it. It is divided by 9 and the remainder is zero. The smallest such natural number is

(A) 15 (B) 3 (C) 6 (D) 1

Solution: Since $11a + 33 = 9a + 2a + 6 + 27$ is a multiple of 9, we need $2a + 6$ must be a multiple of 9. The least number therefore is 6.

Answer: (C)

8. n is a natural number such that $(n + 1)(n + 2)(n + 3) = 201320132013$. Then

(A) n is a two digit number

(B) n is a three digit number

(C) No such n exists

(D) n is a three digits number ending with 3.

Solution: Since $n+1, n+2, n+3$ are consecutive integers, at least one of them must be even and hence their product is also

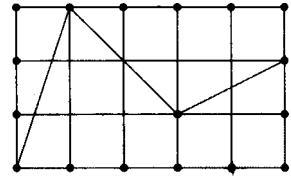
even. But 201320132013 is an odd number. Thus no such n exists.

Answer: (C)

9. What fraction of the rectangle is shaded?

(A) $\frac{1}{2}$ (B) $\frac{17}{30}$ (C) $\frac{5}{8}$ (D) $\frac{8}{15}$

Solution: Total number of small squares is 15. Using the formula for area of a triangle as half the length of the base times the height, we see than the shaded area is $\frac{3}{2}+2+2+2+1=\frac{17}{2}$ times the area of the small square. Hence the ratio is $\frac{\frac{17}{2}}{15}=\frac{17}{30}$. Answer: (B)



10. The first and the third digit of a five digit number $d6d41$ are the same. If the number is divisible by 9, the sum of its digits is
(A) 18 (B) 36 (C) 25 (D) 27

Solution: Since the number is divisible by 9, it follows that $d+6+d+4+1$ is a multiple of 9. Thus $2d+11$ is a multiple of 9 and hence $2(d+1)$ is a multiple of 9. Thus $d=8$ and sum of digits is 27.

Answer: (D)

PART - B

11. Good Shepherd high school has 1584 students. $\frac{4}{9}$ of the students are girls. The number of boys more than the girls in the school is ———.

Solution: The school strength is 1584. Of these the number of girls equals $1584 \times \frac{4}{9} = 704$. Thus number of boys $= 1584 - 704 = 880$. Hence there are $880 - 704 = 176$ boys more than the girls.

This can also be seen as follows: Since the girls are $\frac{4}{9}$ of the students, boys are $\frac{5}{9}$ of the strength. Hence there are $\frac{1}{9} \times 1584 = 176$ more boys than girls.

Answer: 176.

12. There is a lengthy rope. Ram cuts one third of the rope. From the remaining Rahim cuts one fifth. The total length of the rope both cut is 861 metres. The length of the original uncut rope is ———.

Solution: Cut length is $\frac{1}{3} + \frac{2}{15} = \frac{7}{15}$. Given $\frac{7}{15} = 861$ m. Hence the original length is $861 \times \frac{15}{7} = 1845$ m.

Answer: 1845 m

13. Square papers of black and white are arranged as shown

■ ■ □ □ ■ □ ■ ■ □ □ ■ □ ■ ■ □ ···.

There are totally 80 squares. The number of white squares is ———

Solution: The pattern is 2 black squares, 2 white squares, 1 black square, 1 white square, 2 black squares, 2 white squares and so on. Thus in the first ten squares there are 5 white and 5 black squares and the pattern continues. Hence out of 80 squares, 40 should be white squares.

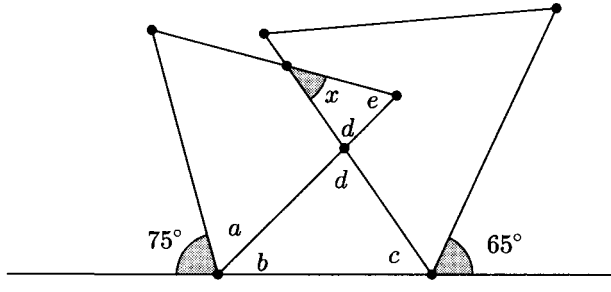
Answer: 40.

14. A company promised Govind Rs. 21,000 and a gift for working 2 years. But Govind left the job in 16 months and got Rs. 12,500 and the gift as compensation. The gift is worth of Rs. ———.

Solution: For 24 months amount agreed is Rs.21000 and a gift. For 16 months amount due in rupees is $21000 \times \frac{16}{24} = 14000$. Since he received Rs. 12500, the value of the gift is Rs. $14000 - 12500 = 1500$.

Answer: Rs.1500

15. Two equilateral triangles overlap as in the figure. The value of the angle x is ———.



Solution: Let the angles be as marked. We know $a = e = 60°$. Hence $b = 180° - 75° - 60° = 45°$. Similarly, $c = 180° - (65° + 60°) = 55°$. Hence $d = 180° - (45° + 55°) = 80°$ and $x = 180° - (80° + 60°) = 40°$.

Answer: 40°

16. x and y are the digits of a two digit number xy. x is greater than y by 3. When this two digit number is divided by the sum of its digits the quotient is 7 and the remainder is 3. The sum of the digits of the two digit number is ———.

Solution: Let the two digit number be $10x + y$. Given that $x = y + 3$ and $10x + y = 7(x + y) + 3$. Substituting for x we get $11y + 30 = 14y + 24$ and hence $y = 2$ and consequently, $x = 5$. The sum of the digits of the number is hence 7.

Answer: 7.

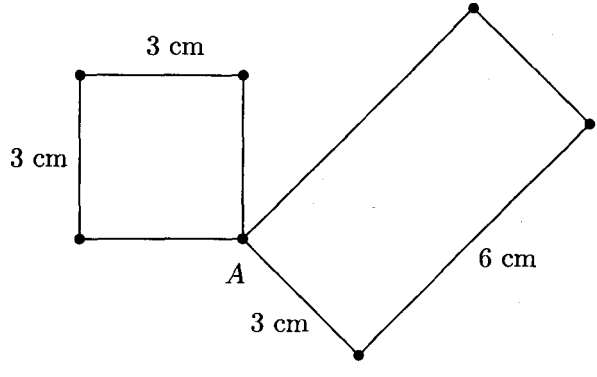
17. If the square roots of the natural numbers 1 to 200 are written down, the number of whole numbers among them is ———.

Solution: Square numbers from 1 to 200 are

$$1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144, 169, 196.$$

and hence there are 14 such numbers.
Answer: 14.

18. Two ants start at A and walk at the same speed, one along the square and the other along the rectangle. The minimum distance (in cm) any one must cover before they meet again is ———.



Solution: They will meet again only at A. The perimeter of the square is 12 cms and that of the rectangle is 18 cms.

Hence the minimum distance they need to cover before they meet again at A is 36 cms.

Answer: 36 cm

19. When 26 is divided by a positive integer N, the remainder is 2. The sum of all possible values of N is ———.

Solution: Since 26 leaves a remainder 2 when divided by N, 24 is divisible by N. Thus the possible values of N are the divisors of 24. There are 8 positive divisors for 24: $1, 2, 3, 4, 6, 8, 12, 24$ but of these 1, 2 do not leave a remainder 2 when dividing 26. Hence the possible numbers are $3, 4, 6, 8, 12, 24$ and their sum is 57.

Answer: 57

20. If

$$\left(1+\frac{1}{2}\right)\left(1+\frac{1}{4}\right)\left(1+\frac{1}{6}\right)\left(1+\frac{1}{8}\right)\left(1+\frac{1}{10}\right) \times \left(1-\frac{1}{3}\right)\left(1-\frac{1}{5}\right)\left(1-\frac{1}{7}\right)\left(1-\frac{1}{9}\right) = 1+\frac{1}{n},$$

then the value of n is ———.

Solution: Given

$$\frac{3}{2}\times\frac{5}{4}\times\frac{7}{6}\times\frac{9}{8}\times\frac{11}{10}\times\frac{2}{3}\times\frac{4}{5}\times\frac{6}{7}\times\frac{8}{9} = 1+\frac{1}{n}.$$

Rearranging the left hand side, we get

$$\frac{3}{2}\times\frac{2}{3}\times\frac{5}{4}\times\frac{4}{5}\times\frac{7}{6}\times\frac{6}{7}\times\frac{9}{8}\times\frac{8}{9}\times\frac{11}{10} = \frac{11}{10}.$$

Hence $\dfrac{11}{10} = 1+\dfrac{1}{10} = 1+\dfrac{1}{n}$ and $n = 10$.

Answer: 10.

SCREENING TEST – KAPREKAR CONTEST
NMTC at SUB JUNIOR LEVEL
VII & VIII Standards

PART - A

1. If $a+b+c=0$ where a, b, c are non zero real numbers, then the value of $(a^2-bc)^2-(b^2-ca)(c^2-ab)$ is

   (A) 1 (B) abc (C) $a^2+b^2+c^2$ (D) 0

   Solution 1: Using $a+b+c=0$, we have

   $$(a^2-bc)^2=((b+c)^2-bc)^2=(b^2+c^2+bc)^2$$

   Now,

   $$\begin{aligned}(b^2-ca)(c^2-ab) &= (b^2+c(b+c)))(c^2+b(b+c)) \\ &= (b^2+c^2+bc)^2\end{aligned}$$

   Thus,

   $$(a^2-bc)^2-(b^2-ca)(c^2-ab)=(b^2+c^2+bc)^2-(b^2+c^2+bc)^2=0$$

   Answer: (D)

   Solution 2:

   $$\begin{aligned}&(a^2-bc)^2-(b^2-ca)(c^2-ab) \\ &\quad=a^4+b^2c^2-2a^2bc-(b^2c^2-c^3a-ab^3+a^2bc) \\ &\quad=a^4+ab^3+ac^3-3a^2bc \\ &\quad=a(a^3+b^3+c^3-3abc) \\ &\quad=a(a+b+c)(a^2+b^2+c^2-ab-bc-ca) \\ &\quad=0.\end{aligned}$$

2. In the adjoining incomplete magic square the sum of all numbers in any row or column or diagonals is a constant value. The value of x is

17				
23	5			
	6	13	20	x
10			21	
11				9

(A) 18 (B) 24 (C) 22 (D) 16

Solution: Let the missing number in the first column, third row be y. Sum of diagonal elements is $17+5+13+21+9 = 65$ and hence sum of numbers in the first column is also 65. Thus $17 + 23 + y + 10 + 11 = 65$ and hence $y = 4$. Now, sum of numbers in third row $= 4 + 6 + 13 + 20 + x = 65$ and hence we have $x = 22$.

Answer: (C)

3. Aruna, Bhanu and Rita have some amount of money. The ratio of the money of Aruna to that of Bhanu is 7:15 and the ratio of the money of Bhanu and Rita is 7:16. If Aruna has Rs. 490, the amount of money Rita has is (in Rupees)

(A) 1500 (B) 1600 (C) 2400 (D) 3600

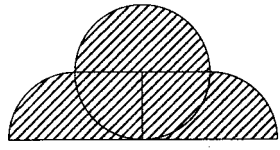
Solution: Let Aruna have x Rs., Bhanu y Rs. and Rita z Rs. Given that $x : y = 7 : 15$ and $y : z = 7 : 16$ or $\frac{x}{y} = \frac{7}{15}$ and hence $y = \frac{15 \times x}{7} = \frac{15 \times 490}{7} = 1050$. Also $\frac{y}{z} = \frac{7}{16}$ and hence $z = \frac{16 \times y}{7} = \frac{16 \times 1050}{7} = 16 \times 150 = 2400$ Rs.

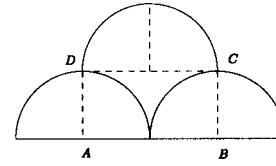
An easier way is as follows: We have $x : y = 7 : 15$ and $y : z = 7 : 16$ and hence $x : y : z = 49 : 105 : 240$. Since $x = 490$, it follows that $z = 2400$.

Answer: (C)

4. In the figure below, all the three semicircles have equal radius of 1 unit. The area of the shaded portion is

(A) $\pi + 2$ (B) 5 (C) $\frac{3\pi}{2} + 1$ (D) 4





Solution: Total area = Area of rectangle $ABCD + 4\times$ (area of a quadrant) $= 2 + \pi$ square units.

Answer: (A)

5. The sum of three different prime numbers is 40. What is the difference between the two biggest ones among them?

(A) 8 (B) 12 (C) 20 (D) 24

Solution: Since the sum is even, all the three primes can not be odd primes. Hence the sum of two of the primes is $40 - 2 = 38$. Thus the other two primes must be 7 and 31 (of all the positive odd numbers whose sum is 38, the rest of the pairs have at least one composite number). Thus the required difference is 24.

Answer: (D)

6. Peter has written down four natural numbers. If he chooses three of his numbers at a time and adds up each triple, he obtains totals of 186, 206, 215 and 194. The largest number Peter has written is

(A) 93 (B) 103 (C) 81 (D) 73

Solution: Let the numbers be a, b, c, d. Given

$$\begin{aligned} a+b+c &= 186 \\ a+c+d &= 206 \\ b+c+d &= 215 \\ a+b+d &= 194 \end{aligned}$$

Adding the above, we get

$$3(a+b+c+d) = 801$$

and hence $a+b+c+d = 267$. The largest among the numbers is therefore $d = 267 - 186 = 81$.

Answer: (C)

7. There are four non-zero numbers x, y, z and u.
If $x = y - z$, $y = z - u$, $z = u - x$, then the value of

$$\frac{x}{y} + \frac{y}{z} + \frac{z}{u} + \frac{u}{x}$$

is equal to

(A) 1 (B) $\frac{1}{2}$ (C) 0 (D) $-\frac{1}{2}$

Solution: Adding the given equations we get $x + y + z = y - z + z - u + u - x$ and hence $z = -2x$. Now, $x = y - z$ and substituting for z, we get $x = -y$. We have, $y = z - u$ and this yields $u = -x$.

$$\frac{x}{y} + \frac{y}{z} + \frac{z}{u} + \frac{u}{x} = -1 + \frac{1}{2} + 2 - 1 = \frac{1}{2}.$$

Answer: (B)

8. The natural numbers from 1 to 20 are listed below in such a way that the sum of each adjacent pair is a prime number.
$20, A, 16, 15, 4, B, 12, C, 10, 7, 6, D, 2, 17, 14, 9, 8, 5, 18, E$. The number D is

(A) 1 (B) 3 (C) 11 (D) 13

Solution: The missing numbers in the list are $1, 3, 11, 13, 19$. Since $20 + 1, 20 + 13, 20 + 19$ are not primes, we have either $A = 3$ or $A = 11$. If $A = 11$, then $A + 16 = 27$ is not a prime. Thus $A = 3$.

B must be one of $1, 11, 13, 19$. B can not be 11 ($4 + 11$ is not a prime) or 13 ($13 + 12$ is not a prime). Thus $B = 1$ or 19. If $B = 1$, then C must be one of $11, 13, 19$. C can not be 11 ($11 + 10$ is not a prime) and can not be 13 ($12 + 13$ is not a prime). Thus $C = 19$. This leaves only $11, 13$ for D. D can not be 13 ($13 + 2$ is not a prime) and hence $D = 11$ and $E = 13$.

If $B = 19$, then C must be one of $1, 11, 13$. Since C can not be 11 or 13, this gives $C = 1$. As before, we can conclude that $D = 11$. Hence even though B and C are not uniquely determined, D is uniquely determined and $D = 11$.

Answer: (C)

9. How many ordered pairs of natural numbers (a, b) satisfy $a + 2b = 100$?

(A) 33 (B) 49 (C) 50 (D) 99

Solution: $a = 100 - 2b$. Clearly $1 \leq b \leq 49$.

Answer: (B)

10. The value of

$$(\sqrt{5} + \sqrt{6} + \sqrt{7})(\sqrt{5} + \sqrt{6} - \sqrt{7}) \times (\sqrt{5} - \sqrt{6} + \sqrt{7})(\sqrt{6} + \sqrt{7} - \sqrt{5})$$

is (A) $3\sqrt{210}$ (B) 210 (C) $4\sqrt{210}$ (D) 104

Solution:

$$(\sqrt{5}+\sqrt{6}+\sqrt{7})(\sqrt{5}+\sqrt{6}-\sqrt{7})$$
$$= (\sqrt{5}+\sqrt{6})^2 - (\sqrt{7})^2 = 4 + 2\sqrt{30}$$

$$(\sqrt{5}-\sqrt{6}+\sqrt{7})(\sqrt{6}+\sqrt{7}-\sqrt{5})$$
$$= (\sqrt{7}+(\sqrt{5}-\sqrt{6}))(\sqrt{7}-(\sqrt{5}-\sqrt{6}))$$
$$= (\sqrt{7})^2 - (\sqrt{5}-\sqrt{6})^2 = -4 + 2\sqrt{30}$$

Given expression $= (4+2\sqrt{30})(-4+2\sqrt{30}) = 104.$

Answer: (D)

11. The least number which, when divided by 52 leaves a remainder 33, when divided by 78 leaves a remainder 59 and when divided by 117 leaves a remainder 98 is

(A) 553 (B) 293 (C) 468 (D) 449

Solution: Let N be the number with the stated properties. First observe that $52-33=19$, $78-59=19$ and $117-98=19$. Since N leaves a remainder 33 when divided by 52, it follows that $N+19$ is divisible by 52. Similarly, $N+19$ is also divisible by 78 and 117. Thus the least number $N+19$ is the least common multiple of 52, 78 and 117 and hence $N+19=468$ and $N=468-19=449$ will satisfy the given conditions.

Answer: (D)

12. A sum of money is divided between two persons in the ratio 3:5. If the share of one person is Rs. 2000 more than that of the other, then the sum of money is (in rupees)

(A) 6000 (B) 8000 (C) 10,000 (D) 12,000

Solution: The shares of the two persons are $3x, 5x$. Given that the difference $2x$ is 2000 Rs. Thus $x = 1000$ and the sum of money is $8x = 8000$ Rs.

Answer: (B)

13. Two numbers are respectively 20% and 50% more than a third number. What percent of the second number is the first number?

(A) 70% (B) 30% (C) 80% (D) 60%

Solution 1: Let a, b be the two numbers and let x be the third number. Given that $a = 1.2x$ and $b = 1.5x$. Thus

$$\frac{a}{b} = \frac{1.2x}{1.5x} = \frac{4}{5}.$$

Hence a is 80% of b.

Answer: (C)

Solution 2: If the third number is assumed to be 100, then the first number is 120 and the second number is 150. Hence the ratio is $\frac{120}{150} = \frac{4}{5}$ and the first number is 80% of the second number.

14. There are some toys. One third of them are sold at a profit of 15%, one fourth of the total are sold at a profit of 20% and the rest for 24% profit. The total profit is Rs. 3200. The total price of the toys is (in rupees)

(A) 32000 (B) 64000 (C) 16000 (D) 48000

Solution: Let the total price of the toys be x Rs. One third of the toys are sold at a profit of 15%. Hence the profit obtained in this sale is $\frac{x}{3} \times 0.15$. One fourth sold at a profit of 20% fetches a profit of $\frac{x}{4} \times 0.20$. and the rest sold at 24% gives a profit of

$$\left(1 - \frac{x}{3} - \frac{x}{4}\right) \times 0.24$$

The total profit therefore is

$$\frac{x}{3} \times \frac{15}{100} + \frac{x}{4} \times \frac{20}{100} + \left(1 - \frac{x}{3} - \frac{x}{4}\right) \times \frac{24}{100} = 3200.$$

Simplifying, we get $x = 16000$ Rs.

Answer: (C)

15. The radius of a circle is increased by 4 units and the ratio of the areas of the original and the increased circle is 4:9. The radius of the original circle is

(A) 6 (B) 4 (C) 12 (D) 8

Solution: Let R be the radius of the original circle. Given that

$$\frac{\pi R^2}{\pi(4+R)^2} = \frac{4}{9}$$

$$\begin{aligned} 9R^2 - 4(4+R)^2 &= 0 \\ \{(3R - 2(4+R)\}\{(3R + 2(4+R)\} &= 0 \\ (R-8)(5R+8) &= 0 \end{aligned}$$

Since R can not be negative, we have $R = 8$.

This can also be seen as follows:

$$R^2 : (4+R)^2 = 4 : 9 \Rightarrow R : (4+R) = 2 : 3$$

Thus $2(4+R) = 3R$ and $R = 8$.

Answer: (D)

PART - B

16. A natural number less than 100 has remainder 2 when divided by 3, remainder 3 when divided by 4 and remainder 4 when divided by 5. The remainder when the number is divided by 7 is ———.

Solution: The number is 1 less than a multiple of 3, 1 less than a multiple of 4 and 1 less than a multiple of 5. Hence the number is 1 less than any multiple of 3, 4 and 5. The least such number is $3 \times 4 \times 5 - 1 = 59$ and the next such number 119 is more than 100. Thus the required number is 59. When this is

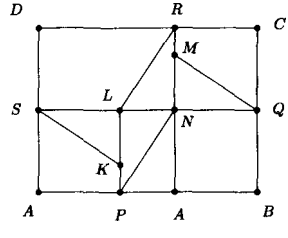
divided by 7, the remainder is 3.
Answer: 3.

17. The units digit of $(2013)^{2013}$ is ————.

Solution: Observe first that the units digit of 2013^{2013} is the same as the units digit of 3^{2013}.
The various powers of 3 are $3^1 = 3$, $3^2 = 9$, $3^3 = 27$, $3^4 = 81$, $3^5 = 243$, ... and hence their units digits are $3, 9, 7, 1, 3, 9, \ldots$. From this repeating pattern of units digits, we see that if n is of the form $1 + 4k$ for some integer k, then the units digit of 3^n is 3. Since $2013 = 1 + 503 \times 4$, it follows that the units digit of 3^{2013} is 3.

Answer: 3.

18. In the figure $ABCD$ is a rectangle with sides 24 units, 32 units. SKL, PLN, QNM, RNL are congruent right angled triangles. The area of each triangle is ————.



Solution:
S, L, N, Q are collinear. Let $LN = LK = NM = x$. $SL = QN = \dfrac{32 - x}{2} = 16 - \dfrac{x}{2} = PL = RN$.
But $PL + RN = AD = 24$ and hence $x = 8$.

The base of each triangle is 8 and height is 12. Thus the area is 48 square units.

Answer: 48 sq. units.

19. The number of natural numbers n for which $\dfrac{(n+2)(n+2)}{(n-3)}$ is a natural number is ————.

Solution:

$$\begin{aligned}\frac{(n+2)(n+2)}{n-3} &= \frac{n^2+4n+4}{n-3} \\ &= \frac{n(n-3)+7n+4}{n-3} \\ &= n+\frac{7n+4}{n-3} \\ &= n+\frac{7(n-3)+25}{n-3} \\ &= n+7+\frac{25}{n-3}\end{aligned}$$

$n-3$ must divide 25. The possibilities are $n-3 = \pm 1, \pm 5, \pm 25$. Thus $n = 2, 4, 8, 28$. When $n = 2$, the given fraction is -4 and hence not a natural number. Thus $n = 4, 8, 28$.

Answer: 3.

20. The product of two natural numbers a and b divides 48. a, b are not relatively prime to each other. The number of pairs (a, b) where $1 < a + b < 48$, $(a \neq b)$ is ————.

Solution: Since ab divides 48, ab must be one of $2, 3, 4, 6, 8, 12, 16, 24, 48$.

Since (a, b) are not relatively prime, ab can not be $2, 3$ or 6.

$$(a,b) = \begin{cases} (2,2) \\ (2,4),(4,2) \\ (2,6),(6,2) \\ (4,4),(2,8),(8,2) \\ (2,12),(12,2),(6,4),(4,6) \\ (2,24),(4,12),(6,8),(24,2),(12,4),(8,6). \end{cases}$$

There are 18 pairs. Since $a \neq b$ reject $(2,2),(4,4)$. Hence there are 16 pairs.

Answer: 16.

21. The value of the expression

$$\left(\frac{a-a^{-2}}{a^{\frac{1}{2}}-a^{-\frac{1}{2}}}-\frac{1-a^{-2}}{a^{\frac{1}{2}}+a^{-\frac{1}{2}}}-a^{\frac{1}{2}}\right)$$

is $\frac{2}{K^3}$ where $a=(2013)^2$. The value of K is ———.

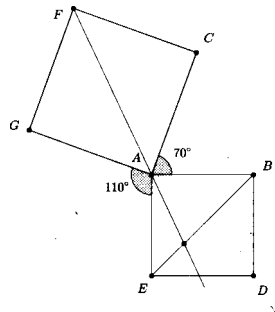
Solution: Given expression

$$\begin{aligned}
&= \frac{(a^3-1)\sqrt{a}}{a^2(a-1)}-\frac{\sqrt{a}(a^2-1)}{(a+1)a^2}-\sqrt{a}\\
&= \frac{(a-1)(a^2+a+1)\sqrt{a}}{a^2(a-1)}-\frac{\sqrt{a}(a+1)(a-1)}{(a+1)a^2}-\sqrt{a}\\
&= \frac{(a^2+a+1)\sqrt{a}}{a^2}-\frac{\sqrt{a}(a-1)}{a^2}-\sqrt{a}\\
&= \frac{\sqrt{a}}{a^2}\left[a^2+a+1-a+1\right]-\sqrt{a}\\
&= \frac{\sqrt{a}}{a^2}\left[a^2+2\right]-\sqrt{a}\\
&= \frac{\sqrt{a}a^2+2\sqrt{a}-\sqrt{a}a^2}{a^2}\\
&= \frac{2\sqrt{a}}{a^2}=\frac{2}{(\sqrt{a})^3}=\frac{2}{(2013)^3} \text{ since } a=(2013)^2.
\end{aligned}$$

Hence the value of K is 2013.

Answer: 2013.

22. AB and AC are two straight line segments enclosing an angle 70°. Squares $ABDE$ and $ACFG$ are drawn outside the angle BAC. The diagonal FA is produced to meet the diagonal EB in H. then $\angle EAH =$ ———.



Solution: Angle at A $= 360° = \angle GAE + 90° + 70° + 90°$ and hence $\angle GAE = 110°$. FAH is a straight line and and $\angle FAG = 45°$. Hence

$$180° = \angle FAG + \angle GAE + \angle EAH \quad (H)$$
$$= 45° + 110 + \angle EAH$$

It follows that $\angle EAH = 25°$.
Answer: $25°$.

23. In the pentagon $ABCDE$, $\angle D = 2\angle B$ and the other angles are each equal to half the sum of the angles $\angle B$ and $\angle D$. The largest interior angle of the pentagon is ————.

Solution: Given that $\angle D = 2\angle B$, and $\angle A + \angle C + \angle E = \frac{3}{2}(\angle B + \angle D)$.

Total of the interior angles equals

$$\frac{5}{2}(\angle B + \angle D) = \frac{3}{2}(3\angle B) = \frac{15}{2}\angle B$$

Since the sum of interior angles in a pentagon is $540°$, it follows that $\frac{15}{2}\angle B = 540°$ and $\angle B = 72°$. So the angles are $108°, 72°, 108°, 144°, 108°$. So, the largest interior angle is $144°$.

Answer: $144°$.

24. The ratio of a two digit number and the sum of its digits is 4:1. If the digit in the units place is 3 more than the digit in the tens place, then the number is ————.

Solution: Let $10a + b$ be the two digit number. Given that $\frac{10a+b}{a+b} = \frac{4}{1}$ and hence $10a + b = 4a + 4b$ or $2a = b$. Also given than $b = a + 3$ and hence $a = 3$ and $b = 6$. The number

is 36.

Answer: 36.

25. There is a famine in a place. But there is sufficient food for 400 people for 31 days. After 28 days 280 of them left the place. Assuming that each person consumes the same amount of food per day, the number of days for which the rest of the food would last for the remaining people is ————.

Solution: Let F be the quantity of food consumed by one person per day. Then there was $400 \times 31F = 12400F$ units of food to begin with. Out of this, in 28 days, the food consumed by 400 people is $400 \times 28F = 11200F$. The remaining food is hence $12400F - 11200F = 1200F$. The remaining people is $400 - 280 = 120$. Hence the remaining food will be sufficient for $\dfrac{1200F}{120F} = 10$ days.

Answer: 10.

26. A triangle whose sides are integers has a perimeter 8. The area of the triangle is ————.

Solution: Let a, b, c be the sides of the triangle. We have $a + b + c = 8$. Since a, b, c are positive integers and sum of two sides of a triangle must be more than the third side, the only possibility is $3, 3, 2$. Thus the triangle is isosceles with base 2 and height $\sqrt{3^2 - 1} = 2\sqrt{2}$. Hence the area equals $\dfrac{1}{2} \times 2 \times 2\sqrt{2} = 2\sqrt{2}$ square units.

Answer: $2\sqrt{2}$.

27. If

$$(10^{2013} + 25)^2 - (10^{2013} - 25)^2 = (\sqrt{10})^n$$

then the value of n is ————.

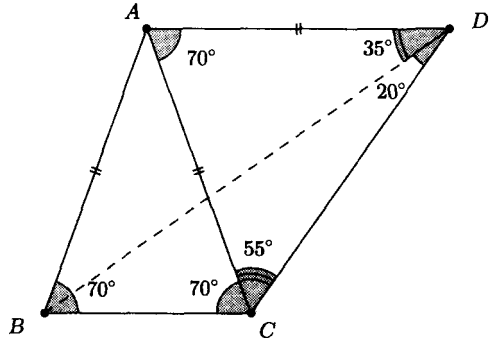
Solution: Since $(a + b)^2 - (a - b)^2 = 4ab$,

$$(10^{2013} + 25)^2 - (10^{2013} - 25)^2 = 4 \times 10^{2013} \times 25 = (10)^{2015}$$

Hence $\frac{n}{2} = 2015$ and $n = 4030$.

Answer: 4030.

28. ABC and ADC are isosceles triangles with $AB = AC = AD$. $\angle BAC = 40°$, $\angle CAD = 70°$. The value of $\angle BCD + \angle BDC =$ ———.



Solution: Since $\angle BAC = 40°$ and $AB = AC$, it follows that $\angle BCA = 70°$. Similarly from the isosceles triangle ACD, we get $\angle ACD = 55°$. Now from the isosceles triangle ABD, since $\angle BAD = 40° + 70°$, $\angle ABD = 35° = \angle ADB$. Since AD is parallel to BC (alternate angles $\angle DAC = 70° = \angle ACB$), $\angle CBD = \angle ADB = 35°$. Hence

$$\angle BCD + \angle BDC = 180° - \angle DBC = 180° - 35° = 145°.$$

Answer: 145°.

29. There are three persons Samrud, Saket and Vishwa. Samrud is twice the age of Saket and Saket is twice the age of Vishwa. Their total ages will be trebled in 28 years. The present age of Samrud is ———.

Solution: Let the present age of Vishwa be x. Saket's age $= 2x$ and Samrud's age $= 4x$. In 28 years, their ages would

be $x + 28, 2x + 28, 4x + 28$. Hence we have

$$(x + 28) + (2x + 28) + (4x + 28) = 3(x + 2x + 4x)$$

Hence $14x = 84$ and $x = 6$. Thus the present age of Samrud is $4x = 24$ years.

Answer: 24 years.

30. The number of three digit numbers of the form $ab5$ which are divisible by 9 is ———.

Solution: Since a number is divisible by 9 only when the sum of its digits is divisible by 9, $a + b + 5$ must be a multiple of 9. Thus $a + b + 5 = 9$ or 18. Consequently, $a + b = 4$ or 13.

When $a + b = 4$ (a, b) is one of

$$(1, 3),\ (3, 1),\ (2, 2),\ (4, 0)$$

and when $a + b = 13$ (a, b) is one of

$$(4, 9),\ (5, 8),\ (6, 7),\ (7, 6),\ (8, 5),\ (9, 4).$$

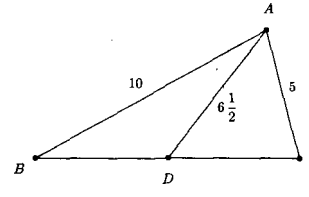
Thus there are 10 such numbers. The numbers are 135, 315, 225, 405, 495, 585, 675, 765, 855, 945.

Answer: 10.

SCREENING TEST – BHASKARA CONTEST
NMTC at JUNIOR LEVEL
IX & X Standards

PART - A

1. Two sides of a triangle are 10 cm and 5 cm in length and the length of the median to the third side is $6\frac{1}{2}$ cm. The area of the triangle is $6\sqrt{x}$ cm^2. The value of x is



(A)12 (B) 13 (C) 14 (D) 15

Solution: Let D be the mid point of the side BC. By Apollonius theorem, $AB^2 + AC^2 = 2(BD^2 + AD^2)$. Hence

$$BD^2 = \frac{1}{2}(10^2 + 5^2) - \left(6\frac{1}{2}\right)^2 = \frac{81}{4}$$

Thus $BD = \frac{9}{2}$ and $BC = 9$. The sides of the triangle are therefore, $a = 9$, $b = 5$, $c = 10$. The semi-perimeter $s = 12$ and the area is given by

$$\sqrt{s(s-a)(s-b)(s-c)} = 6\sqrt{14}$$

Thus $x = 14$.

Answer is (C).

2. x and y are real numbers such that $7^x - 16y = 0$ and $4^x - 49y = 0$, then the value of $(y - x)$ is

(A) $\frac{5}{2}$ (B) $\frac{19}{2}$ (C) $\frac{4115}{2013}$ (D) $\frac{1569}{784}$

Solution: Since $7^x = 4^2y$, $4^x = 7^2y$, $\dfrac{7^x}{4^x} = \dfrac{4^2}{7^2}$ and hence $x = -2$, $y = \dfrac{1}{784}$ and $y - x = \dfrac{1}{784} + 2 = \dfrac{1569}{784}$.

Answer is (D).

3. a, b are positive integers such that

(i) the sum of their squares is S

(ii) the sum of their cubes is C times the sum of the numbers

(iii) $S - C = 28$.

The number of such pairs (a, b) is

(A) 1 (B) 2 (C) 3 (D) 6

Solution: From the given conditions we have

$$a^2 + b^2 - \frac{a^3 + b^3}{a + b} = 28 \quad \Rightarrow ab = 28.$$

The possibilities are

$$(a, b) = (1, 28),\ (2, 14),\ (4, 7),\ (7, 4),\ (14, 2),\ (28, 1).$$

Answer is (D).

4. The number of numbers of the form $30a0b03$ that are divisible by 13, where a, b are digits, is

(A) 5 (B) 6 (C) 7 (D) 0

Solution: We have

$$N = (3000003+10000a+100b) \mod 13 = (6+3a+9b) \mod 13$$

Since N is divisible by 13, $3a+9b = 7 \mod 13$. The maximum value of $3a + 9b$ for digits a, b is 108. Now, $3a + 9b$ must be a multiple of 3 and also leave a remainder 7 when divided by 13. Hence it is equal to 33 or 72. Solving for a, b we obtain the following values for (a, b):

$$(2,3),\ (5,2),\ (8,1),\ (0,8),\ (3,7),\ (6,6),\ (9,5)$$

There seven such numbers. The numbers are:

$$3020303 \quad 3050203 \quad 3080103 \quad 3000803$$
$$3030703 \quad 3060603 \quad 3090503$$

Answer is (C).

5. The number of positive integral values of (x, y) which satisfy the equations $\sqrt[3]{x} + \sqrt[3]{y} = 4,\ x + y = 28$ simultaneously is

(A) 1 (B) 2 (C) 0 (D) 3

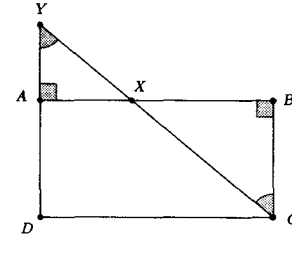
Solution: Cubing both sides of $\sqrt[3]{x} + \sqrt[3]{y} = 4$ we get

$$x + y + 12(xy)^{\frac{1}{3}} = 64$$

Putting $x + y = 28$ in the above, we get $xy = 27$. Hence x, y are roots of $t^2 - 28t + 27 = 0$ and $(x, y) = (1, 27)$ or $(27, 1)$.

Answer is (B).

6. $ABCD$ is a rectangle. Through C a variable line is drawn so as to cut AB at X and DA produced at Y. Then $BX \times DY$ is
(A) twice the area of the rectangle $ABCD$.
(B) equal to the area of the rectangle $ABCD$.



(C) a variable quantity which lies between the area of rectangle $ABCD$ and twice the area of the rectangle $ABCD$.
(D) always a constant less than the area of rectangle $ABCD$.

Solution: Lat $AB = CD = a$, $AD = BC = b$ and $AX = x$. The triangles AXY and BXC are similar. Thus $\frac{b}{AY} =$

$\frac{a-x}{x}$ and $AY = \frac{bx}{a-x}$. Now, $DY = AY + b = \frac{bx}{a-x} + b = \frac{ab}{a-x}$. Also, $BX = a - x$ and hence $BX \times DY = ab$, the area of the rectangle $ABCD$.

Answer is (B).

7. The number of ordered triples (x, y, z) such that x, y, z are primes and $x^y + 1 = z$ is

(A) 0 (B) 1 (C) 2 (D) infinitely many.

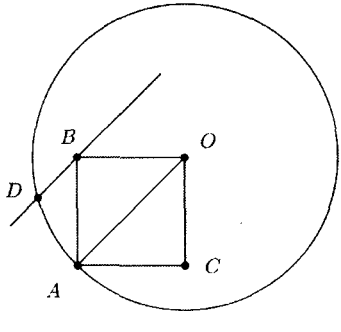
Solution: If y is odd then $x^y + 1$ is divisible by $x + 1$ and z can not be a prime. Thus y must be even and since it is a prime, $y = 2$. It follows that z must be an odd prime. But then $x^y = z - 1$ implies that x is even and hence $x = 2$. Consequently $z = 5$ and the only triple satisfying the given condition is $(2, 2, 5)$.

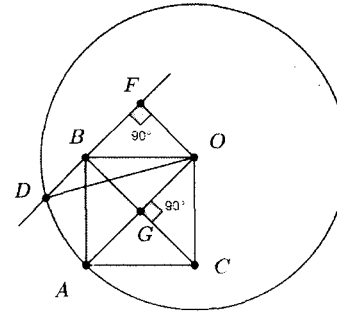
Answer is (B).

8. In the adjoining figure, O is the centre of the circle. $ACOB$ is a square with A on the circle. Through B a line parallel to OA is drawn to cut the circle at D nearer to A. Then $\angle BOD =$

(A) 20° (B) 18° (C) 15° (D) $22\frac{1}{2}°$

Solution: Draw $OF \perp DB$ produced. The diagonals of the





square $OBAC$ intersect at G at right angles. Since $BG = GO$ clearly $BFOG$ is a square.

$$OF = BG = \frac{1}{2}BC = \frac{1}{2}OA = \frac{1}{2}OD.$$

Hence in the triangle DOF, $\sin(\angle ODF) = \frac{1}{2}$ and $\angle ODF = 30°$. Thus $\angle DOF = 60°$ and since $\angle BOF = 45°$, it follows that $\angle BOD = 15°$.

Answer is (C).

9. The number of real solutions of the equation

$$x + \sqrt{x^2 + \sqrt{x^3 + 1}} = 1$$

is

(A) 1 (B) 2 (C) 3 (D) 0

Solution:

$$\begin{aligned} x + \sqrt{x^2 + \sqrt{x^3 + 1}} = 1 \quad &\Rightarrow \quad x^2 + \sqrt{x^3 + 1} = (1 - x)^2 \\ &\Rightarrow \quad \sqrt{x^3 + 1} = 1 - 2x \\ &\Rightarrow \quad x^3 + 1 = 1 - 4x + 4x^2 \\ &\Rightarrow \quad x^3 - 4x^2 + 4x = 0 \\ &\Rightarrow \quad x(x^2 - 4x + 4) = 0 \\ &\Rightarrow \quad x = 0, 2, 2. \end{aligned}$$

When $x = 2$, the given equation is not satisfied. Hence $x = 0$ is the only solution.

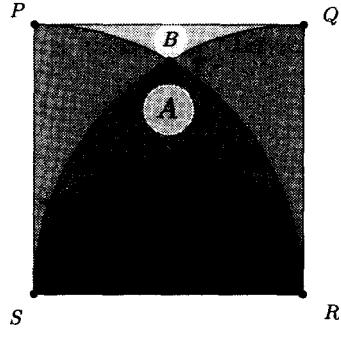
Answer is (A).

10. In the figure below $PQRS$ is a square of side 2 units. PTR and QTS are quadrants of circles of radius 2 units. With SR as diameter, a semicircle is drawn. A, B denote the areas of the portions shaded.

Then $(A - B) =$

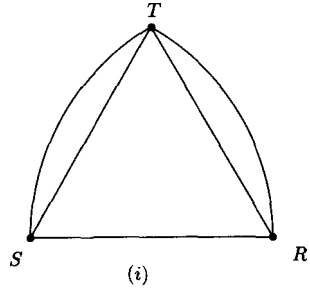
(A) $\frac{3\pi}{2} - 4$ (B) $\frac{\pi - 1}{3}$

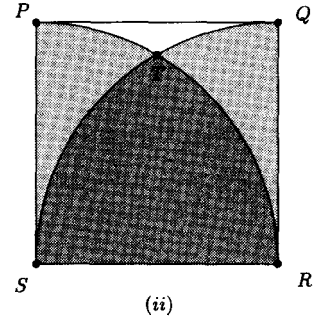
(C) $\frac{3}{2} - \frac{\pi}{4}$ (D) $4 - \pi$



Solution: In the following figure (i), we first calculate the area bounded by the sectors with center S and arc RT and the sector with center R and arc ST.

Area of the sector with center S and arc RT is $\frac{1}{6}\pi 2^2 = \frac{2}{3}\pi$,





since the angle of the sector is $60°$. Similarly, the area of the sector with center R and arc TS is $= \frac{2}{3}\pi$. Hence area of the region bounded by these sectors is

$$\frac{2}{3}\pi + \frac{2}{3}\pi - \sqrt{3} = \frac{4}{3}\pi - \sqrt{3} \qquad (1)$$

where we subtracted the area of the equilateral triangle STR, since it is added twice in adding the areas of the sectors. Since the area of the semi circle with SR as diameter is $\frac{\pi}{2}$, we have

$$A = \frac{5}{6}\pi - \sqrt{3} \tag{2}$$

Now, in figure (ii), area of the quadrant PSR is π and is also the area of the quadrant SRQ. The area bounded by the union of these two quadrants is

$$2\pi - \left(\frac{4}{3}\pi - \sqrt{3}\right) = \frac{2\pi}{3} + \sqrt{3} \tag{3}$$

where we have used (1). The area B can be obtained by subtracting this area from the area of the square. Hence

$$B = 4 - \left(\frac{2\pi}{3} + \sqrt{3}\right) \tag{4}$$

From (2) and (4) we get

$$A - B = \frac{5}{6}\pi - \sqrt{3} - \left(4 - \left(\frac{2\pi}{3} + \sqrt{3}\right)\right) = \frac{3}{2}\pi - 4$$

Answer is (A).

11. a, b, c are digits of a 3-digit number such that $64a + 8b + c = 403$, then the value of $a + b + c + 2013$ is

(A) 2024 (B) 2025 (C) 2034 (D) 2035

Solution: Since $64a + 8b + c$ leaves a remainder c when divided by 8 and 403 leaves a remainder 3 when divided by 8, we conclude $c = 3$. We now have $64a + 8b = 400$ and dividing throughout by 8, we get $8a + b = 50$. Since a, b are numbers less than 10, it follows that $a = 6$ and $b = 2$. Hence $a + b + c + 2013 = 6 + 2 + 3 + 2013 = 2024$.

Answer is (A).

12. What is the sum of the digits of $(9999999999)^3$

(A) 99 (B) 108 (C) 180 (D)199

Solution: The given number is

$$9999999999^3 = (10^{10} - 1)^3.$$

Let us look at the sum of the digits of $(10^n - 1)^3$ for small values of n. Since

$$(10^n - 1)^3 = 10^{3n} - 3 \cdot 10^{2n} + 3 \cdot 10^n - 1$$

we have

n	$10^{3n} - 3 \cdot 10^{2n}$	$3 \cdot 10^n - 1$	$(10^n - 1)^3$	Sum of digits
1	700	29	729	18
2	970000	299	970299	36
3	997000000	2999	997002999	54
4	999700000000	29999	999700029999	72

We observe that each row has one more 9 at the beginning and one more 9 at the end and one more zero in the middle than the previous row. Thus the sum increases by 18 every stage.

For $n = 10$, the sum of digits is therefore 180.

Answer is (C).

13. The number of three digit numbers which are divisible by 3 and have the additional property that the sum of their digits is 4 times their middle digit is

(A) 7 (B) 4 (C) 11 (D) 10

Solution: Let $N = 100a + 10b + c$ be one such number. Since N is a multiple of 3, it follows that the sum of the digits $a+b+c$ is also a multiple of 3. Again, given that the sum of the digits is 4 times the middle digit. Thus sum of the digits is also a multiple of 4. Hence $a + b + c$ must be divisible by 12. The maximum value of $a + b + c$ is 27. The only possibilities are

$a+b+c=12$ and $a+b+c=24$.
When $a+b+c=12$, we have $b=3$ and $a+c=9$. This yields nine numbers:

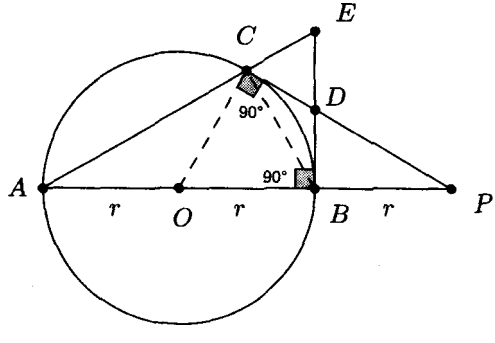
$$138,\ 237,\ 336,\ 435,\ 534,\ 633,\ 732,\ 831,\ 930$$

When $a+b+c=24$, $b=6$ and $a+c=18$. This gives only one number: 969. Thus we have 10 such numbers.

Answer is (D).

14. In the figure below, AB is a diameter of a circle. AB is produced to P such that $BP =$ radius of the circle. PC is a tangent to the circle. The tangent at B and AC produced cut at E. Then $\triangle CDE$ is

(A) isosceles with $EC = ED$ (B) isosceles with $EC = CD$
(C) equilateral (D) a scalene triangle



Solution: Let O be the center of the circle. In triangle PCO, $\angle OCP = 90°$ and $\sin\angle OPC = \dfrac{r}{2r} = \dfrac{1}{2}$. Thus $\angle OPC = 30°$ and $\angle COP = 60°$. Hence triangle COB is equilateral and

$BC = r$.

Also in triangle ACB, $\angle ACB = 90°$ and $\sin \angle CAB = \frac{r}{2r} = \frac{1}{2}$ and hence $\angle CAB = 30°$. Now CP is a tangent to the circle and $\angle PCB =$ angle in the segment CAB of the circle $= \angle CAB = 30°$. Consequently, $\angle PCE = \angle BCE - \angle PCB = 90° - 30° = 60°$.

Also since $\angle OPC = 30°$, we have $\angle CDE = \angle PDB = 60°$. Thus in the triangle CDE we have $\angle ECD = \angle CDE = 60°$ and hence it is equilateral.

Answer is (C).

15. Nine numbers are written in ascending order. The middle number is the average of the nine numbers. The average of the five largest numbers is 68 and the average of the five smallest numbers is 44. The sum of all numbers is

(A) 560 (B) 504 (C) 112 (D) 122

Solution: Let S denote the sum of all numbers.

The middle number is $\frac{S}{9}$. The sum of the five largest number is $5 \times 68 = 340$. The sum of the five smallest numbers is $5 \times 44 = 220$. If we add these two, we would have added the middle numbers twice. Thus

$$S + \frac{S}{9} = 340 + 220 = 560$$

Hence $S = 504$.

Answer is (B).

PART – B

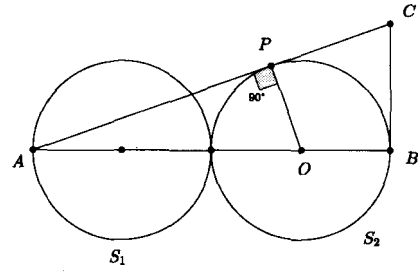
16. The least value of the positive integer n such that $(n+20) + (n+21) + (n+22) + \cdots + (n+100)$ is a perfect square is ———.

Solution: There are 81 terms and it is an arithmetic progression. Hence the sum $= 81(n + 60)$. If this is a perfect

square, $n + 60$ must be a perfect square. The least value for n is 4.

Answer: 4.

17. In the figure below two equal circles S_1, S_2 of radii 2 units each touch each other. AB is the common diameter. The tangent at B meets the tangent from A to the circle S_2 at C as shown. If $BC = K\sqrt{2}$ then the value of K is ———.



Solution: From triangle AOP, $AP^2 = AO^2 - OP^2 = 6^2 - 2^2 = 32$. Let $BC = x$. Since triangles AOP and ACB are similar, we have

$$\frac{BC}{AB} = \frac{OP}{AP}$$

and hence

$$\frac{x}{8} = \frac{2}{4\sqrt{2}}$$

Thus $BC = x = 2\sqrt{2}$ and $K = 2$.

Answer: 2.

18. When the number $33333^2 + 22222$ is written as a single decimal number, the sum of its digits is ———.

Solution: Let us look at some simple cases:

$$3^2 + 2 = 11,\ 33^2 + 22 = 1111,\ 333^2 + 222 = 111111,\ \ldots$$

Hence we see that

$$33333^2 + 22222 = 1111111111$$

and the sum of the digits is 10.

Answer: 10.

19. The number of three digit numbers such that the product of their digits is a prime number is ———.

Solution: Let the number be abc.

Since $a \times b \times c$ is a prime, we should have two of them as 1 each and the third, a prime. Since a, b, c are digits, we can have only 2,3,5,7 for the prime number among a, b, c. If the prime is 2, we have three numbers: $112, 121, 211$. Similarly we have three numbers each for the other primes. Thus the number of numbers such that the product of their digits is a prime is 12.

Answer: 12.

20. The number of real values (x, y) for which

$$2^{x+1} + 3^y = 3^{y+2} - 2^x$$

is ———.

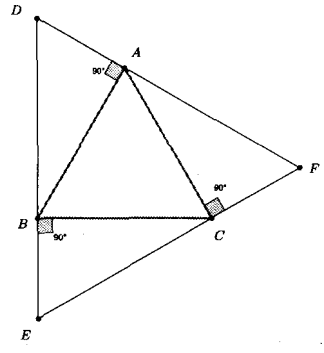
Solution: Since $2^{x+1} + 2^x = 3^{y+2} - 3^y$ we have $2^x(2+1) = 3^y(3^2 - 1)$ or $3 \times 2^x = 8 \times 3^y$ and hence $2^{x-3} = 3^{y-1}$.

This is possibly only when $x - 3 = 0$, $y - 1 = 0$, that is, $(x, y) = (3, 1)$ is the only solution.

Answer: 1.

21. In the figure below, $\triangle ABC$ is equilateral. AD, BE and CF are respectively perpendicular to AB, BC and AC. Then $\dfrac{\text{Area of } \triangle DEF}{\text{Area of } \triangle ABC} =$ ———.

Solution: Clearly, triangle DEF is equilateral. Let $2a$ be the side of the triangle ABC. Then from $\dfrac{AC}{AF} = \cos 30°$, we get $AF = \dfrac{4a}{\sqrt{3}}$. Also from $\dfrac{AD}{AB} = \tan 30°$, we get $AD = \dfrac{2a}{\sqrt{3}}$.



Thus $DF = DA + AF = \dfrac{6a}{\sqrt{3}}$ and the area of the triangle DEF is given by

$$\sqrt{3}\left(\frac{3a}{\sqrt{3}}\right)^2 = 3\sqrt{3}a^2 = 3 \times \text{ area of triangle } ABC.$$

Hence

$$\frac{\text{Area of } \triangle DEF}{\text{Area of } \triangle ABC} = 3.$$

Answer: 3.

22. If $f(x) = ax + b$ and $f(f(f(x))) = 27x + 26$ then $a + b =$ ———.

Solution: $f(f(f(x))) = a^3x + a^2b + ab + b = 27x + 26$. Hence $a = 3$ and $b = 2$ and $a + b = 5$.

Answer: 5

23. The eight digits 6, 5, 5, 4, 4, 3, 2 and 1 are used to form two 3-digit numbers and one 2-digit number. The largest possible sum of these numbers is ———.

Solution To get the largest sum, the individual numbers must be as large as possible. Thus the numbers in the hundreds digit and in the tens digit must be chosen as large as possible. Using

6 and 5 in the hundreds place of the two three digit numbers and 5 and 4 in the tens place, we get the numbers 653,542,41. The sum of these is $653 + 542 + 41 = 1236$.

Answer: 1236.

24. $a \neq 0$, $b \neq 0$. The number of real number pairs (a, b) which satisfy the equation $a^4 + b^4 = (a + b)^4$ is ———.

Solution: We have

$$(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4.$$

Given equation reduces to

$$4a^3b + 6a^2b^2 + 4ab^3 = 0$$

or

$$2ab(2a^2b + 3ab + 2b^2) = 0$$

Since $ab \neq 0$, we have

$$2a^2b + 3ab + 2b^2 = 0$$

$$2\left(\frac{a}{b}\right)^2 + 3\left(\frac{a}{b}\right) + 2 = 0$$

This quadratic in $\frac{a}{b}$ has discriminant equal to $9 - 16 = -7$, negative and hence the quadratic has no real roots. Thus there are no such pairs of real numbers.

Answer: 0.

25. The number of integers greater than 2 and less than 70 that can be written as a^b (where $b \neq 1$) is ———.

Solution: Listing out the various powers of numbers less than 70, we have

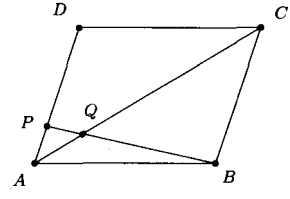
$$2^2 = 4,\ 2^3 = 8,\ 2^4 = 16,\ 2^5 = 32,\ 2^6 = 64,\ 3^2 = 9,\ 3^3 = 27,$$

$$5^2 = 25,\ 6^2 = 36,\ 7^2 = 49.$$

We have a total of 10 numbers. Powers of 4 and 8 need not be considered because they are covered in the powers of 2.

Answer: 10.

26. $ABCD$ is a parallelogram P is a point on AD such that $\dfrac{AP}{AD} = \frac{1}{2013}$. Q is the point of intersection of AC and BP. Then $\dfrac{AQ}{AC} =$ ———.



Solution: Clearly, triangles AQP and CQB are similar. Thus

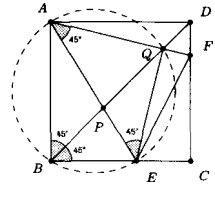
$$\frac{AQ}{QC} = \frac{AP}{BC} = \frac{AP}{AD} = \frac{1}{2013}$$

Hence we have

$$\frac{AQ}{AC} = \frac{AQ}{AQ + QC} = \frac{1}{2014}$$

Answer: $\dfrac{1}{2014}$.

27. $ABCD$ is a square. E and F are points respectively on BC and CD such that $\angle EAF = 45°$. AE and AF cut the diagonal BD at P, Q respectively. Then $\dfrac{\text{Area of } \triangle AEF}{\text{Area of } \triangle APQ} =$ ———



Solution: Since $\angle QAE = \angle QBE = 45°$, $AQEB$ is cyclic. Thus $\angle AEQ = 45°$ and AEQ is a right angled isosceles

triangle and $\frac{AE}{AQ} = \sqrt{2}$. Similarly, $\frac{AF}{AP} = \sqrt{2}$.

$$\frac{\text{Area of } \triangle AEF}{\text{Area of } \triangle APQ} = \frac{\frac{1}{2}AE \cdot AF \sin 45}{\frac{1}{2}AP \cdot AQ \sin 45} = \sqrt{2} \cdot \sqrt{2} = 2.$$

Answer: 2.

28. m, n are natural numbers. The number of pairs (m, n) for which $m^2 + n^2 + 2mn - 2013m - 2013n - 2014 = 0$ is ———.

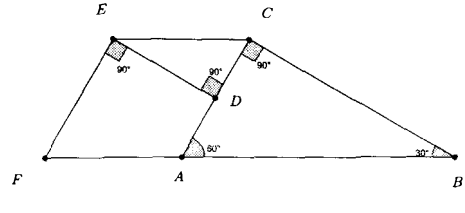
Solution: Since $(m+n)^2 - 2013(m+n) - 2014 = 0$, we have $(m+n+1)(m+n-2014) = 0$. Thus either $m+n = -1$ or $m+n = 2014$. Since m, n are natural numbers, $m+n \neq -1$. Thus $m+n = 2014$ and the possible pairs (m, n) are

$$(1, 2013), (2, 2012), \ldots, (2013, 1)$$

and there are 2013 such pairs.

Answer: 2013.

29. In the adjoining figure BAC is a $30° - 60° - 90°$ triangle with $AB = 20$. D is the midpoint of AC. The perpendicular at D to AC meets the line parallel to AB through C at E. The line through E perpendicular to DE meets BA produced at F. If $DF = 5\sqrt{x}$ then $x =$ ———.



Solution: $\triangle ABC$ is a $30° - 60° - 90°$ triangle and $AB = 20$, we have $AC = 10$. Since D is the mid point of AC, it follows that $DC = 5$. $\triangle ECD$ is also a $30° - 60° - 90°$ triangle. Hence $EC = 10$ and since AC has also length 10, it follows that $ACEF$ is a rhombus. Hence

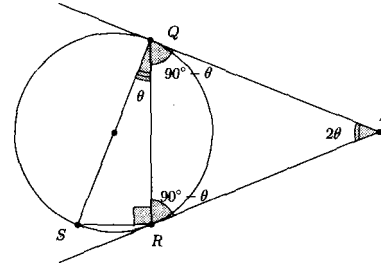
$$DF^2 = DE^2 + EF^2 = (EC^2 - CD^2) + AC^2 = 100 - 25 + 100 = 175$$

Thus $DF = 5\sqrt{7}$ and $x = 7$.

Answer: 7.

30. PR and PQ are tangents to a circle and QS is a diameter.

Then $\dfrac{\angle QPR}{\angle RQS} =$ ———.



Solution: Let $\angle QPR = 2\theta$. Since triangle QPR is isosceles, $\angle PQR = 90° - \theta$ and since QS is a diameter and PQ is a tangent to the circle, $\angle RQS = \theta$. Hence $\dfrac{\angle QPR}{\angle RQS} = \dfrac{2\theta}{\theta} = 2$.

Answer: 2.

SCREENING TEST – RAMANUJAN CONTEST NMTC at INTER LEVEL XI & XII Standards

PART - A

1. The number of non-zero integers x such that $-10 < x < 15$ for which $3x^3 + 7x^2$ is the square of an integer is

   (A) 5 (B) 6 (C) 0 (D) 1

   Solution: We have $x^2(3x + 7) = y^2$. Since x^2 is a square, $(3x+7)$ also must be a square. Let $3x+7 = 3(x+2)+1 = z^2$. Also since $3x+7$ eaves a remainder 1 when divided by 3, $z^2 = 1$ mod 3. Hence $z = 3m \pm 1$ and

   $$3x + 7 = (3m \pm 1)^2 = 9m^2 \pm 6m + 1$$

   Thus $x = 3m^2 \pm 2m - 2$. Since $x < 15$, $|m| < 3$ and substituting values of m we get $x = -2, -1, 3, 6, 14$.

   There are 5 integers in the interval $-10 < x < 15$.

   Answer: (A)

2. The vertices of a triangle ABC are lattice points (that is, points with integer coordinates). Two of its sides have lengths which belong to the set $\{\sqrt{19}, \sqrt{2013}, \sqrt{2014}\}$. The maximum possible area of the triangle is

   (A) 2013 (B) 1007 (C) 2014 (D) $\sqrt{2014}\}$

   Solution: Without loss of generality, assume that the lengths AB, BC belong to the set $\{\sqrt{19}, \sqrt{2013}, \sqrt{2014}\}$.

   Then the area of $\triangle ABC$ is

   $$\frac{1}{2} \cdot AB \cdot BC \sin(\angle BCA) \leq \frac{1}{2}\sqrt{2014}\sqrt{2014}\sin\frac{\pi}{2}$$

Equality holds when exactly two sides have lengths $\sqrt{2014}$ and the angle between them is $\frac{\pi}{2}$.

Answer: (B)

3. The minimum value of the terms of the sequence

$$\sqrt{\frac{7}{6}+\sqrt{\frac{96}{7}}}, \sqrt{\frac{8}{6}+\sqrt{\frac{96}{8}}}, \sqrt{\frac{9}{6}+\sqrt{\frac{96}{9}}}, \cdots, \sqrt{\frac{95}{6}+\sqrt{\frac{96}{95}}}$$

is

(A) 6 (B) 7 (C) 8 (D) 4

Solution: BY AM-GM inequality

$$\sqrt{\frac{n}{6}}+\sqrt{\frac{96}{n}} \geq 2\sqrt{\sqrt{\frac{n}{6}}\sqrt{\frac{96}{n}}} = 2\sqrt{4} = 4.$$

Inequality holds when $\sqrt{\frac{n}{6}} = \sqrt{\frac{96}{n}}$, that is when $n = 24$.

Minimum value the term 4 and it is $\sqrt{\frac{24}{6}}+\sqrt{\frac{96}{24}}$.

Answer: (D)

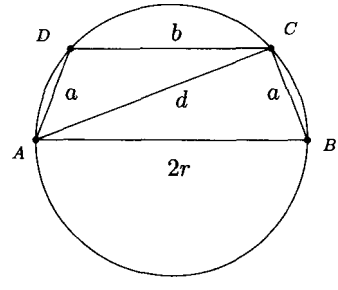
4. $ABCD$ is a cyclic quadrilateral in a circle of radius r. AB is a diameter of the circle. CD is parallel to AB. $CD = b$, $AD = BC = a$. The value of $\frac{2r^2 - a^2}{br}$ is

(A) 1 (B) 2 (C) 3 (D) 4

Solution: By Ptolemy's theorem, in the cyclic quadrilateral $ABCD$, we have

$$AB \cdot CD + AD \cdot BC = AC \cdot BD$$

Thus, $a^2 + 2br = d^2$ where $d = AC$. Since $\angle ACB = 90°$, $a^2 + d^2 = 4r^2$.



Hence $a^2 + 2br = 4r^2 - a^2$ or $a^2 + br = 2r^2$.

It follows that $\dfrac{2r^2 - a^2}{br} = 1$.

Answer: (A)

5. Positive integers a, b, c are chosen so that $a < b < c$ and the system of equations $2x + y = 2013$ and $y = |x - a| + |x - b| + |x - c|$ have exactly one solution. Then the minimum value of c is

(A) 760 (B) 1007 (C) 2013 (D) 2012

Solution: The system has only one solution and hence the two graphs cut at only one point.

Taking into various possibilities for x we have

$$y = \begin{cases} -3x + a + b + c & \text{if} \quad x < a \\ -x - a + b + c & \text{if} \quad a \le x < b \\ x - a - b + c & \text{if} \quad b \le x < c \\ 3x - a - b - c & \text{if} \quad x \ge c. \end{cases}$$

Thus the graph consists of lines with slopes $-3, -1, 1, 3$ and has corners at $(a,\ b + c - 2a), (b,\ c - a)$ and $(c,\ 2c - a - b)$

$y = -2x + 2013$ has slope -2.

If the graphs cut at exactly one point that point must be $(a,\ b+c-2a)$.

$\therefore\ 2013 = 2a+b+c-2a \quad \therefore\ b+c = 2013.$ Given $b < c$.

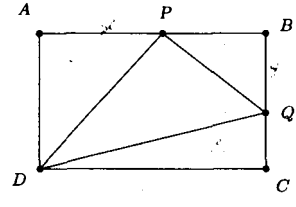
Minimum value of c is 1007.

Answer: (B)

6. $ABCD$ is a rectangle. P and Q are points on AB and BC respectively such that the area of triangle $APD = 5$, area of triangle $PBQ = 4$ and area of triangle $QCD = 3$, all areas in square units. Then the area of the triangle DPQ in square units is

(A) 12 (B) $\frac{20}{3}$ (C) $2\sqrt{21}$ (D) $\sqrt{21}$

Solution: Let the lengths of the sides AB and BC be a and b respectively. Let $AP = x$ and $BQ = y$. Then we have



$$\begin{aligned} xb &= 10 \quad (1)\\ (a-x)y &= 8 \quad (2)\\ (b-y)a &= 6 \quad (3) \end{aligned}$$

From (1) we get

$$x = \frac{10}{b} \tag{4}$$

and from (3) we get

$$y = b - \frac{6}{a} \tag{5}$$

Substituting for x, y in (2), we get

$$\left(a - \frac{10}{b}\right)\left(b - \frac{6}{a}\right) = 8 \tag{6}$$

$$ab + \frac{60}{ab} - 24 = 0 \tag{7}$$

or

$$(ab)^2 - 24(ab) + 60 = 0 \tag{8}$$

Solving (8), we get

$$ab = 12 \pm 2\sqrt{21}$$

Since $ab > 0$, we have $ab = 12 + 2\sqrt{21}$. Now, the area of triangle DPQ equals $12 + 2\sqrt{21} - (3 + 4 + 5) = 2\sqrt{21}$.

Answer: (C)

7. The number of right triangles of integer length sides and the product of the leg lengths is equal to three times the perimeter is

(A) 0 (B) 1 (C) 2 (D) 3

Solution: Let the lengths of the legs of the triangle be a, b and the length of the hypotenuse be c. Then we have $ab = 3(a + b + c)$. Also by Pythogoras theorem, $c^2 = a^2 + b^2$. Substituting for c, we get

$$9c^2 = (ab - 3a - 3b)^2 = 9(a^2 + b^2)$$

Simplifying, we get

$$ab((a - 6)(b - 6) - 18) = 0$$

The solutions are

$$(a, b) = (7, 24),\ (8, 15),\ (9, 12)$$

In the above cases, $c = 25, 17, 15$ respectively. Thus there are three such triangles.

Answer: (D)

8. Let S be the set of ordered triples (x, y, z) of real numbers for which $\log_{10}(x + y) = z$ and $\log_{10}(x^2 + y^2) = z + 1$. a, b are real numbers such that for all ordered triples (x, y, z) in S, we have $x^3 + y^3 = a \cdot 10^{3z} + b \cdot 10^{2z}$. Then the value of $(a + b)$ is

(A) $\frac{15}{2}$ (B) $\frac{29}{2}$ (C) 15 (D) $\frac{39}{2}$

Solution: $x + y = 10^z$ and $x^2 + y^2 = 10^{z+1}$. Hence

$$xy = \frac{10^{2z} - 10 \cdot 10^z}{2}$$

Also, $(x + y)^3 = 10^{3z}$ and $x^3 + y^3 = (x + y)^3 - 3xy(x + y)$. Hence

$$x^3 + y^3 = -\frac{1}{2} \cdot 10^{3z} + 15 \cdot 10^{2z}$$

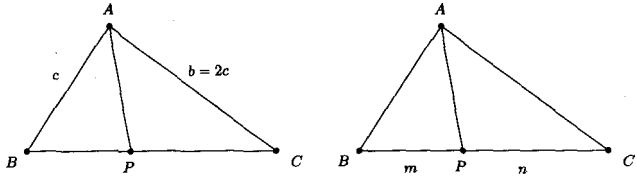
Now,

$$a + b = -\frac{1}{2} + 15 = \frac{29}{2}.$$

Answer: (B)

9. The internal bisector of $\angle A$ of a triangle ABC meets BC at P and $b = 2c$. If $9AP^2 + 2a^2 = k \cdot c^2$, then k is

(A) 8 (B) 3 (C) 19 (D) 18



Solution: We will use the generalized Apollonius theorem: If in a triangle ABC, P is point on the side BC such that $BP : PC = m : n$, then

$$m \cdot AC^2 + n \cdot AB^2 = m \cdot PC^2 + n \cdot BP^2 + (m + n) \cdot AP^2$$

Using bisector theorem, we get $\frac{BP}{PC} = \frac{c}{2c} = \frac{1}{2}$. Thus $BP : PC = 1 : 2$. Also $BP = \frac{1}{3}a$ and $PC = \frac{2}{3}a$. By Apollonius

theorem we have

$$2AB^2 + AC^2 = 2 \cdot \left(\frac{a}{3}\right)^2 + \left(\frac{2a}{3}\right)^2 + (2+1)AP^2$$

and hence

$$6c^2 = \frac{6a^2}{9} + 3AP^2 = \frac{2a^2}{3} + 3AP^2$$

Thus $9AP^2 + 2a^2 = 18c^2$ and $k = 18$.

Answer: (D)

10. The product of 5 odd primes is a five digit number of the form $ab0ab$ where a, b are digits and the hundreds digit is zero. The number of such numbers is

(A) 0 (B) 9 (C) 13 (D) 18

Solution: The 5-digit number is

$$\begin{aligned} 10^4a + 10^3b + 10^2 \times 0 + 10a + b &= 10^3(10a + b) + (10a + b) \\ &= (10a + b)(1001). \end{aligned}$$

Now $1001 = 11 \times 7 \times 13$ and three of the five primes in the product are $7, 11$ and 13. Also $(10a + b)$ must be a two digit number and is a product of two primes.

If the least of the primes is 3, then the other prime can be one of $5, 7, 11, 13, 17, 19, 23, 29, 31$ and there are 9 such numbers.

If the least of the primes is 5, the other prime is one of $5, 7, 11, 13, 17, 19$ and there are 6 such numbers.

If the least of the primes is 7, the other prime can be $7, 11, 13$ and there are 3 numbers.

Thus we have a total of 18 numbers.

Answer: (D)

11. The number of perfect square divisors of the product $1!\ 2!\ 3!\ 4!\ \cdots\ 9!$ is

(A) 583 (B) 615 (C) 627 (D) 672

Solution:

$$\begin{aligned} 1!\ 2!\ 3!\ 4!\ \cdots\ 9! &= (1)(1.2)(1.2.3)\cdots(1.2.3.4\cdots 9) \\ &= 1^9.2^8.3^7.4^6.5^5.6^4.7^3.8^2.9^1 \\ &= 2^{30}.3^{13}.5^5.7^3 \end{aligned}$$

The perfect square divisors are of the form $2^{2a} \cdot 3^{2b} \cdot 5^{2c} \cdot 7^{2d}$ where $0 \leq a \leq 15,\ 0 \leq b \leq 6,\ 0 \leq c \leq 2,\ 0 \leq d \leq 1$.

Thus there are $(16)(7)(3)(2) = 672$ such divisors.

Answer: (D)

12. Srilekha, Priyanka, Vidya and Vishwa bought a gift for their classmate's birthday. The gift is with someone of the four. Each makes a statement. One statement is false and the other three are true.
Srilekha : I do not have the gift and Vidya does not have the gift.

Priyanka: I do not have the gift and Srilekha does not have the gift.

Vidya : I do not have the gift and Priyanka does not have the gift.

Vishwa : I do not have the gift and Srilekha does not have the gift.

The gift is with

(A) Srikekha (B) Priyanka (C) Vidya (D) Vishwa

Solution: The following gives the truth table of the 4 statements depending on who has the gift.

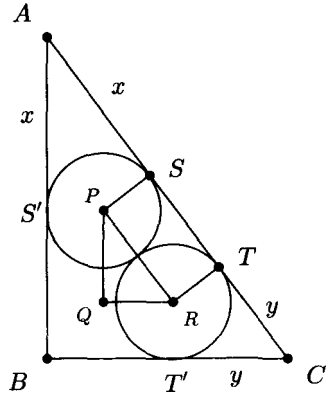
	Srilekha	Priyanka	Vidya	Vishwa
Srilekha has the gift	False	False	True	False
Priyanka has the gift	True	False	False	True
Vidya has the gift	False	True	False	True
Vishwa has the gift	True	True	True	False

The only case in which there is exactly one false statement occurs if Vishwa has the gift.

Answer: (D)

13. In the figure below the triangle is a 3 - 4 - 5 sides triangle. Two equal circles are placed as in the figure. The radius of each circle is

(A) $\frac{4}{5}$ (B) $\frac{5}{8}$ (C) $\frac{6}{11}$ (D) $\frac{5}{7}$



Solution: Let r be the radius of the circles, $AS = AS' = x$, $CT = CT' = y$. Clearly, $PR = 2r$, $PQ = 4 - x - r$, $QR =$

$3-y-r$, $x+y+2r=5$. Since the triangle PQR is similar to ABC we have

$$\frac{PR}{5}=\frac{QR}{3}=\frac{PQ}{4}$$

and hence

$$\begin{aligned}\frac{2r}{5} &= \frac{3-y-r}{3}=\frac{4-x-r}{4}\\ &= \frac{(3-y-r)+(4-x-r)}{3+4}=\frac{7-(x+y+2r)}{7}\\ &= \frac{7-5}{7}=\frac{2}{7}\end{aligned}$$

Thus $r=\frac{5}{7}$.

Answer: (D)

14. The number of positive integer pairs (m,n) where n is odd, that satisfy $\frac{1}{m}+\frac{4}{n}=\frac{1}{12}$ is

(A) 1 (B) 2 (C) 3 (D) 11

Solution: We have

$$12n+48m=mn \tag{1}$$

Clearly 4 divides mn. Since n is odd it follows that 4 divides m. Let $m=4k$. From (1), we get

$$3n+48k=kn \tag{2}$$

Hence n divides $48k$ and since n is odd, n must divide $3k$. Let $3k=nt$ where t is an integer. Then substituting in (2), we get

$$9+48t=nt \tag{3}$$

From (3), t divides 9 and hence $t=1,3,9$ and $n=57,51,49$ respectively. Corresponding values of m are obtained from (1) as $76,204,588$. Thus

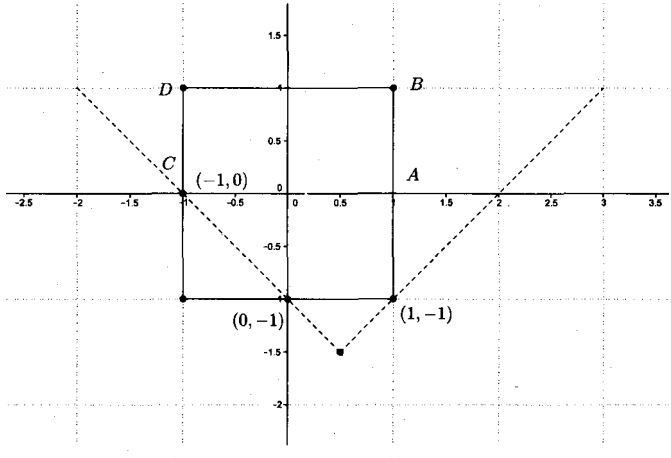
$$(m,n)=(76,57),\ (204,51),\ (588,49)$$

and there are 3 pairs.

Answer: (C)

15. The number of points common to the two graphs whose equations are $||x|-|y||+|x|+|y|=2$ and $2y=|2x-1|-3$ is

(A) 1 (B) 2 (C) 3 (D) 4



Solution: If $|x| \geq |y|$ then $||x|-|y||+|x|+|y|=2$ simplifies to $|x|=1$ and $|y| \leq 1$. This consists of the segments AB and CD of the square shown in the figure. Proceeding similarly, we see that the graph of the curve $||x|-|y||+|x|+|y|=2$ consists of the four sides of the square.

The graph of the curve $2y=|2x-1|-3$ is shown in dotted line in the figure. Hence there are three points of intersection, $(-1,0),(0,-1),(1,-1)$.

Answer: (C)

PART - B

16. When simplified, the value of

$$\frac{(10^4+324)(22^4+324)(34^4+324)(46^4+324)(58^4+324)}{(4^4+324)(16^4+324)(28^4+324)(40^4+324)(52^4+324)} \text{ is } ___.$$

Solution: $324 = 18^2$. All factors of the form $(a^4 + 18^2)$

$$a^4 + 18^2 = (a^2 + 18)^2 - 36a^2 = (a^2 + 18 + 6a)(a^2 + 18 - 6a).$$

Let N be the given expression.

All factors are of the form $\dfrac{(10 + 12k)^4 + 18^2}{(4 + 12k)^4 + 18^2}$

$$\begin{aligned} N &= \prod_{k=0}^{4} \frac{(10 + 12k)^4 + 18^2}{(4 + 12k)^4 + 18^2} \\ &= \prod_{k=0}^{4} \frac{[(10 + 12k)^2 + 18 + 6(10 + 12k)][(10 + 12k)^2 + 18 - 6(10 + 12k)]}{[(4 + 12k)^2 + 18 + 6(4 + 12k)][(4 + 12k)^2 + 18 - 6(4 + 12k)]} \\ &= \prod_{k=0}^{4} \frac{(144k^2 + 312k + 178)(144k^2 + 168k + 58)}{(144k^2 + 168k + 58)(144k^2 + 24k + 10)} \\ &= \prod_{k=0}^{4} \frac{144k^2 + 312k + 178}{144k^2 + 24k + 10} \end{aligned}$$

Numerator $= 144k^2 + 312k + 178 = 144(k+1)^2 + 24(k+1) + 10$.
Hence if we set $g(k) = 144k^2 + 24k + 10$, then

$$N = \prod_{k=0}^{4} \frac{g(k+1)}{g(k)} = \frac{g(5)}{g(0)} = \frac{144 \cdot 5^2 + 312 \cdot 5 + 178}{10} = 373.$$

Answer: 373.

17. The least positive integer a such that the equation

$$\cos^2 \pi(a - x) - 2\cos \pi(a - x) + \cos \frac{3\pi x}{2a} \cdot \cos\left(\frac{\pi x}{2a} + \frac{\pi}{3}\right) + 2 = 0$$

has a real root is ———.

Solution: The given equation can be written as

$$(\cos \pi(a - x) - 1)^2 + \left(1 + \cos \frac{3\pi x}{2a} \cdot \cos\left(\frac{\pi x}{2a} + \frac{\pi}{3}\right)\right) = 0$$

Since both terms are positive and their sum is zero, it follows that each of them is equal to zero. Hence

$$\cos(\pi(a-x)) - 1 = 0 \tag{1}$$

$$1 + \cos\frac{3\pi x}{2a} \cdot \cos\left(\frac{\pi x}{2a} + \frac{\pi}{3}\right) = 0 \tag{2}$$

From the second equation, we obtain

$$\cos\frac{3\pi x}{2a} = \pm 1 \tag{3}$$

$$\cos\left(\frac{\pi x}{2a} + \frac{\pi}{3}\right) = \mp 1 \tag{4}$$

From (3), we get

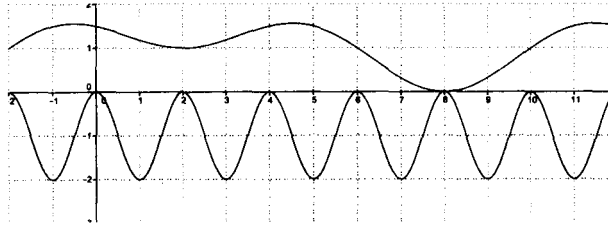
$$\frac{3x}{2n} = a \tag{5}$$

where n is an integer and from (4) we get

$$\frac{x}{2(m - \frac{1}{3})} = a \tag{6}$$

where m is an integer.

When $x = 2kn$ for some integer k, (5) is satisfied and the value of a is $3k$. When $x = 2l\left(m - \frac{1}{3}\right)$ where l is an integer, (6) is satisfied and the value of a is $2l$. Hence a must be a multiple of 2 and 3 and the least value of a is 6.



The figure shows the graphs of the equations (1) and (2) when $a = 6$. The curves meet at $x = 8$.

18. The quadrilateral $ABCD$ is inscribed in a circle. The diagonals AC and BD intersect at Q. The sides DA and CB are produced to meet at P. Given the $CD = CP = DQ$, the value of $\angle CAD =$ ———.

Remark The given conditions appear inconsistent (under the conditions, the quadrilateral $ABCD$ becomes degenerate).

19. Dividing a three digit number by the number obtained from it by swapping its first and last digits we get 3 as the quotient and the sum of the digits of the original number as the remainder. The number of such 3-digit numbers is ———.

Solution: Let $100a+10b+c$ be the three digit number. From the given condition we get

$$100a + 10b + c = 3(100c + 10b + a) + a + b + c$$

Simplifying we get $32a = 100c+7b$. Since $a \leq 9$, it follows that $c \leq 2$. Clearly, $c \neq 0$. If $c = 1$, $32a - 100$ must be positive and a multiple of 7. But $32a - 100 = (28a - 98) + 2(2a - 1)$ and hence $2a - 1$ must be a multiple of 7. $a = 4$ is the only possibility. When $c = 1$ and $a = 4$, we have $b = 4$ and 441 is a number that satisfies the given conditions.

When $c = 2$, $32a - 200$ must be positive and a multiple of 7. We see that $a = 8$ and $b = 8$ in this case. Hence 882 is another number satisfying the given conditions. Thus there are only two numbers 441 and 882.

Answer: 2.

20. The number of pairs (a, b) of real numbers such that the zeros of the polynomials $(6x^2 - 24x - 4a)$ and $(x^3 + ax^2 + bx - 8)$ are all non-negative real numbers is ———.

Solution: Let α, β be the roots of $6x^2 - 24x - 4a = 0$. Then

$$\alpha + \beta = 4 \tag{1}$$

$$\alpha\beta = -\frac{2}{3}a \tag{2}$$

Let x_1, x_2, x_3 be the roots of $x^3 + ax^3 + bx - 8 = 0$. Then

$$x_1 + x_2 + x_3 = -a \tag{3}$$

$$x_1x_2 + x_2x_3 + x_3x_1 = b \tag{4}$$

$$x_1x_2x_4 = 8 \tag{5}$$

Since the roots are non-negative, $4\alpha\beta \leq (\alpha + \beta)^2$. This gives $a \geq -6$. From (3), $x_1+x_2+x_3 \leq 6$. By $A.M-G.M$ inequality,

$$8 = x_1x_2x_3 \leq \left(\frac{x_1 + x_2 + x_3}{3}\right)^3 \leq 8 \tag{6}$$

which implies that $x_1 = x_2 = x_3 = 2$ (when equality holds in $A.M - G.M$ inequality, all the numbers are equal). Thus $a = -6$ and $b = 12$. Thus $(-6, 12)$ is the only ordered pair of reals satisfying the given conditions.

Answer: 1.

21. The number of pairs of positive integers (m, n) such that $m \times n = \gcd(m, n) + \text{lcm}(m, n)$ is ——.

Solution: The left hand side and the lcm (m, n) are both divisible by m and hence m divides $\gcd(m, n)$. Also since $\gcd(m, n) \leq m$, we have $\gcd(m, n) = m$. Similarly, $\gcd(m, n) = n$. Thus $m = n$.

The given equation can now be written as $m^2 = m + m = 2m$ and hence $m(m - 2) = 0$. Since $m > 0$, it follows that $m = 2$. Thus only the pair $(2, 2)$ satisfies the given condition.

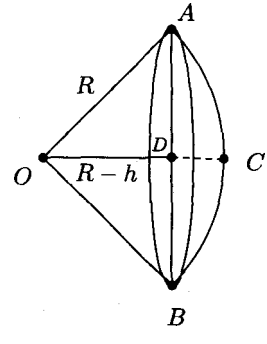
Answer: 1.

22. Two spheres of radii 3 cm and 4 cm, whose centres are 5 cm apart cut each other. The volume common to both the spheres is given by $\frac{a}{b}\pi$ cm^3 where $\frac{a}{b}$ is the fraction with the lowest numerator and denominator, then the value of $\frac{a}{b}$ is ——.

Solution: We need the formula for the volume of a spherical cap (the slice of a solid sphere cut by a plane).

If we have a spherical cap of height h (see Figure *Spherical Cap*), the volume of the cap is given by

$$\pi h^2 \left(R - \frac{h}{3} \right)$$



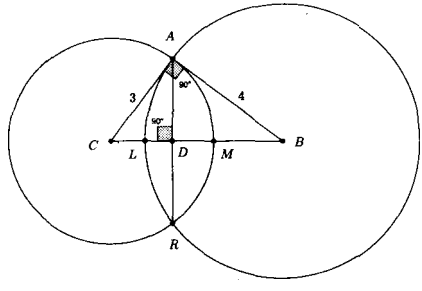
Spherical Cap

The figure *Intersecting Spheres* represents a vertical cross section of the spheres through the line of centres.

Clearly $\angle BAC = 90°$. By the similarity of the triangles ABC and DBA, we have

$$\frac{AB}{BC} = \frac{BD}{AB} = \frac{4}{5}$$

and hence $BD = \frac{16}{5}$ cm. $DL = BL - BD = 4 - \frac{16}{5} = \frac{4}{5}$ cm



Intersecting Spheres

and $MD = CM - CD = CM - (BC - BD) = 3 - \left(5 - \frac{16}{5}\right) = \frac{6}{5}$ cm.

Volume of the segment AMR in cubic cms equals

$$\pi\left(\frac{6}{5}\right)^2\left(3 - \frac{1}{3} \times \frac{6}{5}\right) = \frac{468}{125}\pi$$

and the volume of the segment ALR in cubic cms equals

$$\pi\left(\frac{4}{5}\right)^2\left(4 - \frac{4}{5} \cdot \frac{1}{3}\right) = \frac{896}{375}\pi$$

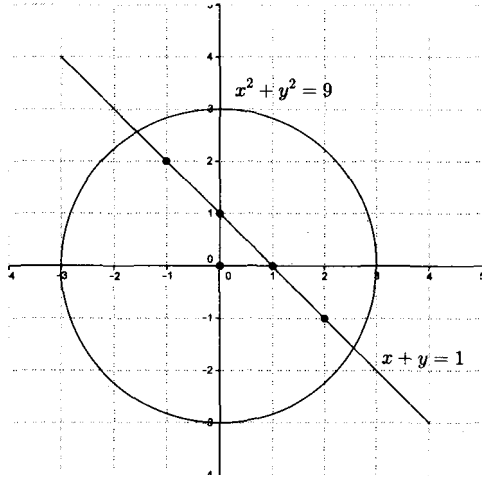
Common volume equals

$$\left(\frac{468}{125} + \frac{896}{375}\right)\pi = \frac{92}{15}\pi$$

Answer: $\frac{92}{15}$.

23. The number of points with integer coordinates on the line $x + y = 1$ which are located inside the circle with centre $(0, 0)$ and radius 3 is ——.

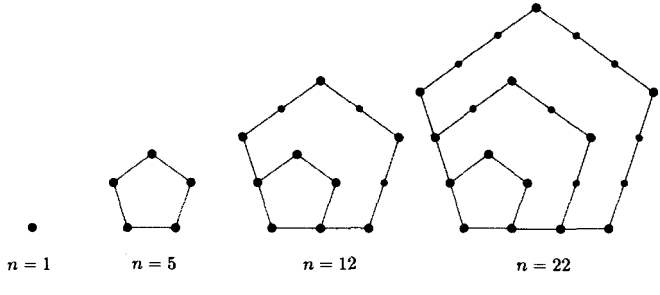
Solution: Plotting the line $x + y = 1$ and the circle $x^2 + y^2 = 9$ we see that there are four points –



$(-1, 2)$, $(0, 1)$, $(1, 0)$, $(2, -1)$ that lie on the line and also lie inside the circle.

Answer: 4.

24. Look at the pattern shown in the Figure Pentagonal Numbers.



Pentagonal Numbers

The numbers 1, 5, 12, 22, . . . are called Pentagonal numbers. If the 2013^{th} Pentagonal number is $2013K$, then K is ——.

Solution: Let P_n be the nth pentagonal number and F_n denote the corresponding nested pentagon figure. Note that the figure F_n contains $n-1$ nested pentagons. The F_{n+1} is obtained by adding 3 segments S_1, S_2, S_3 each containing $n+1$ points. Since the segments have one of the end points as common point, the total number of points added is $3(n+1)-2 = 3n+1$. Thus we have

$$P_{n+1} = P_n + 3n + 1 \tag{1}$$

Writing (1) as $P_{n+1} - P_n = 3n + 1$ and summing over $n = 1, 2, \ldots, 2012$, we get

$$P_{2013} - P_1 = 3(1 + 2 + \cdots + 2012) + 2012 = 3018 \times 2013 + 2012$$

Hence $P_{2013} = 1 + 3018 \times 2013 + 2012 = 3019 \times 2013$, and $K = 3019$.

Answer: 3019.

Remark The $n-$ th pentagonal number is given by

$$P_n = \frac{n(3n-1)}{2}$$

The proof can be easily obtained from (1).

25. The number of natural numbers $n \leq 2013$ for which $2^n - 1$ is divisible by 7 is ——.

Solution: Let us examine the first few numbers. $2^1 - 1 = 1$, $2^2 - 1 = 3, 2^3 - 1 = 7$. Hence $2^3 - 1$ is a multiple of 7. Since

$$x^k - 1 = (x-1)(x^{k-1} + x^{k-2} + \cdots + 1)$$

we have

$$2^{3k} - 1 = (2^3 - 1)((2^3)^{k-1} + \cdots + 1)$$

is also a multiple of 7. Hence all integers of the form $2^{3k}-1$ are multiples of 7. Since $2013 = 3 \times 671$, there are 671 numbers $n \leq 2013$ such that $2^n - 1$ is a multiple of 7.

We show that when n is not divisible by 3, $2^n - 1$ is not a multiple of 7.

$$2^{3k+1} - 1 = 2 \cdot 2^{3k} - 1 = 2 \cdot (7\ell + 1) - 1 = 2 \cdot 7\ell + 1$$

$$2^{3k+2} - 1 = 4 \cdot 2^{3k} - 1 = 4 \cdot (7\ell + 1) - 1 = 4 \cdot 7\ell + 3$$

Thus when n is not a multiple of 3, $2^n - 1$ is not a multiple of 7.

Answer: 671.

26. A fruit basket contains 4 bananas, 5 oranges and 6 apples. The number of ways a person can make a selection of the fruits from among the fruits in the basket is ——.

Solution: Zero or more bananas can be selected from 4 bananas $= (4 + 1) = 5$ ways. Similarly the oranges in 6 ways and the apples in 7 ways.

The number of ways of selecting zero or more fruits from the basket is $= 5 \times 6 \times 7 = 210$.

Excluding the selection of zero fruits, the number of ways is 209.

Answer: 209.

27. a, b, c are positive integers such that $a + b + c = 2013$. Given that $a!b!c! = m10^n$ where m, n are integers and m is not divisible by 10, the smallest value of n is ——.

Solution: Noting that n is the number of factors of 5 in the product $a!b!c!$ and $2013 < 5^5$, none of a, b, c can be divisible

by a power of 5 larger than 4. Thus

$$n = \sum_{i=1}^{4} \left[\frac{a}{5^i}\right] + \sum_{i=1}^{4} \left[\frac{b}{5^i}\right] + \sum_{i=1}^{4} \left[\frac{c}{5^i}\right]$$

$[x]$ is the greatest integer function.

It is easy to see that for any three positive real numbers x, y, z,

$$[x] + [y] + [z] \geq [x + y + z] - 2.$$

Hence

$$\begin{aligned} n &\geq \sum_{i=1}^{4} \left(\left[\frac{a+b+c}{5^i}\right] - 2 \right) \\ &= \sum_{i=1}^{4} \left(\left[\frac{2013}{5^i}\right] - 2 \right) \\ &= 402 + 80 + 16 + 3 - 8 \\ &= 493 \end{aligned}$$

The smallest value for n is 493.

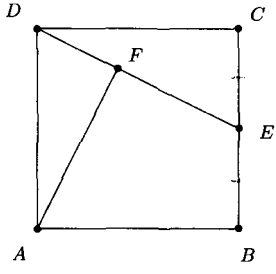
Answer: 493.

28. In the figure (i) below, $ABCD$ is a square. E is the midpoint of CB. AF is drawn perpendicular to DE. If the side of the square is 2013 cm the length of FB is —— cm.

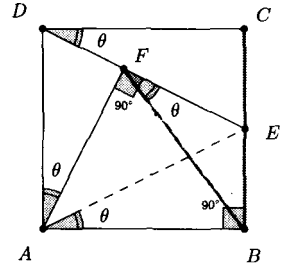
Solution: Join EA, FB (See (ii) above). Let $\angle EAB = \theta$. Clearly, triangles ECD and EBA are congruent. Hence $\angle EDA = \theta$. Also, the quadrilateral $ABEF$ is cyclic and hence $\angle BFE = \angle BAE = \theta$. Now

$$\angle FAB = 90^\circ - \theta = \angle EFA - \angle EFB = \angle BFA$$

Thus triangle ABF is isosceles with $AB = FB$. Hence length of FB is 2013.



(*i*)



(*ii*)

Answer: 2013.

29. The number of primes p for which $(p+2)$ and p^2+2p-8 are both primes is ——.

Solution: If $p=2$, $p+2=4$ is not a prime. Also if $p=3$, $p+2=5$ and $p^2+2p-8=7$ and hence $p=3$ satisfies the condition in the problem.

Let $p>3$. Then p can be written as $p=3k+1$ or $p=3k+2$ for some integer k. If $p=3k+1$, then $p+2=3k+3$ is not a prime. Hence p must be of the form $3k+2$. Now

$$p^2+2p-8=(3k+2)^2+2(3k+2)-8=9(k^2+1)$$

and hence is not a prime. Thus $p=3$ is the only possibility.

Answer:1.

30. $f(x)$ is a linear function. $f(0)=-5$ and $f(f(0))=-15$. The number of values of α for which the solutions to the inequality $f(x)\ f(\alpha-x)>0$ form an interval of length 2 is ——.

Solution: Let $f(x)=ax+b$. Since $f(0)=-5$, we have $b=-5$. Also from $f(f(0))=-15$, we get $a=2$. Hence $f(x)=2x-5$.

$$\begin{aligned} f(x)f(\alpha - x) > 0 &\Leftrightarrow (2x - 5)(2(\alpha - x) - 5) > 0 \\ &\Leftrightarrow (2x - 5)(2\alpha - 5 - 2x) > 0 \\ &\Leftrightarrow \left(x - \frac{5}{2}\right)\left(\alpha - \frac{5}{2} - x\right) > 0 \\ &\Leftrightarrow \left(x - \frac{5}{2}\right)\left(x - \left(\alpha - \frac{5}{2}\right)\right) < 0 \end{aligned}$$

Thus x lies between $\frac{5}{2}$ and $\alpha - \frac{5}{2}$. The length of the interval in which x lies is $|\alpha - 5|$. Since we need $|\alpha - 5| = 2$, $\alpha = 7, 3$ and there are two possible values for α.

Answer: 2.

FINAL – GAUSS CONTEST
NMTC at PRIMARY LEVEL
V & VI Standards

1. There are 4 girls and 2 boys of different ages. The eldest is 10 years old while the youngest is 4 years old. The older of the boys is 4 years older than the youngest of the girls. The oldest of the girls is 4 years older than the youngest of the boys. What is the age of the oldest of the boys?

   Solution: Let the ages of the girls be g_1, g_2, g_3, g_4 and the ages of the boys be b_1, b_2. We can assume that $g_1 < g_2 < g_3 < g_4$ and $b_1 < b_2$. If the eldest is a boy, then $b_2 = 10$ and $g_1 = 10 - 4 = 6$. The youngest must be a boy and thus $b_1 = 4$. The oldest of the girls is 4 years older than the youngest of the boys. Hence $g_4 = 8$. But then $g_1 = 6$, $g_4 = 8$ and $g_1 < g_2 < g_3 < g_4$ is not possible since there is only one number between 6 and 8. Consequently, the eldest must be a girl and $g_4 = 10$. This implies that $b_1 = 6$ and thus the youngest must be a girl aged 4 years. Thus the age of the older boy is 8 years.

2. In the equation $A+M+T+I = 10$. A, M, T, I are all different natural numbers. A is the least. Calculate the maximum and minimum values of $A \cdot M \cdot T \cdot I + A \cdot M \cdot T + A \cdot T \cdot I + M \cdot I \cdot T + M \cdot T \cdot I$ (where $\cdot$ means multiplication. i.e., $A \cdot T \cdot I = A \times T \times I$).

   Solution: Since A, M, T, I are different and sum of them is 10, the only possibility is that they are the numbers $1, 2, 3, 4$ in some order. Since A is the least, we have $A = 1$. Now, since $A = 1$, we have

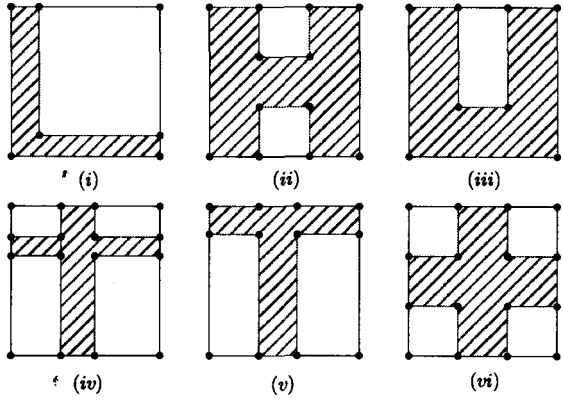
$$A \cdot M \cdot T \cdot I + A \cdot M \cdot T + A \cdot T \cdot I + M \cdot I \cdot T + M \cdot T \cdot I = 3 \cdot M \cdot T \cdot I + (M + I) \cdot T \quad (1)$$

   The product $3M \cdot T \cdot I$ has the same value 72 for all possible assignments of $2, 3, 4$ to M, T, I. Hence the maximum and

minimum of the given expression depends on the maximum and minimum values of $(M+I)\cdot T$. Now this product is maximum when we assign T as the largest number – equal to 4. The value of the product is $(2+3)\times 4 = 20$ and the maximum value of the given expression is 92. Also the product $(M+I)\cdot T$ is minimum when we assign T the least value – equal to 2. The value of the product is $(3+4)\times 2 = 14$ and the minimum value of the given expression is 86.

3. The six squares below are identical. The dimensions of the shaded portions are not known.

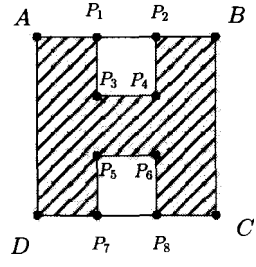
The perimeter of which shaded areas are equal to the



perimeter of the square? Show the calculations clearly and if the perimeter of any shaded area is different from that of the square, state whether it is more or less than the perimeter of the square.

Solution: Let us first consider the area (ii).

Let us name the points as in the figure. The perimeter P of



the shaded area is

$$P = AP_1 + P_1P_3 + P_3P_4 + P_4P_2 + P_2B + BC + CP_8 + P_8P_6 + P_6P_5 + P_5P_7 + P_7D + DA$$

But, clearly, we have $P_3P_4 = P_1P_2$, $P_5P_6 = P_7P_8$,

$$AP_1 + P_3P_4 + P_2B = AP_1 + P_1P_2 + P_2B = AB$$

and

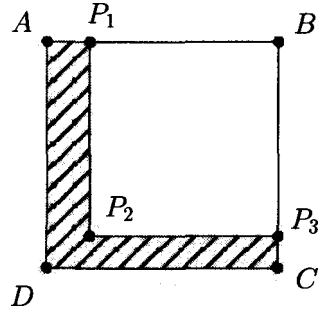
$$CP_8 + P_6P_5 + P_7D = CP_8 + P_8P_7 + P_7D = CD.$$

Thus

$$P = AB + BC + CD + DA + P_1P_3 + P_2P_4 + P_8P_6 + P_7P_5$$

and hence is greater than the perimeter of the square.

Similarly we can show that in the case (*iii*) the perimeter of the shaded area is greater than the perimeter of the square.



Let us consider case (i) now:

Let us name the points as in the figure. The perimeter P of the shaded area is

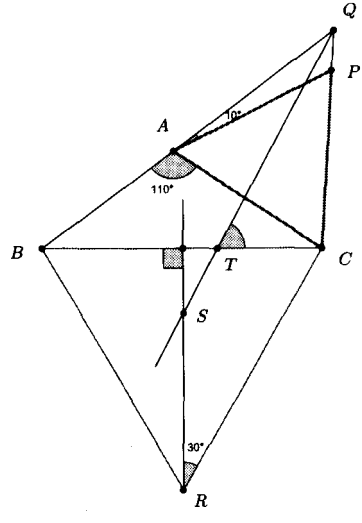
$$P = AP_1 + P_1P_2 + P_2P_3 + P_3C + CD + DA \qquad (2)$$

Clearly, $AP_1 + P_2P_3 = AB$ and $P_1P_2 + P_3C = BC$. Thus, substituting in (2) we get that the perimeter of the shaded area is equal to the perimeter of the square.

One can show similarly that in the cases $(iv), (v)$ and (vi) also the perimeter of the shaded area is equal to the perimeter of the square.

4. ABC is a triangle in which $A = 110°$. $AB = AC$. APC and BRC are equilateral triangles drawn respectively on AC and BC outside the triangle ABC. BA is produced and meets CP produced at Q. The bisectors of $\angle Q$ and $\angle R$ cut at S. Calculate $\angle QSR$. What can you say about the figure $SRCQ$?

Solution (1): We have, $\angle B = \angle C = 35°$. In $\triangle QTC$,



$\angle QTC = 180° - [35° + 25° + 60°] = 60° = \angle TCR$. Thus QT is parallel to CR.

$\angle SQC = 30°$, since SR bisects $\angle CRB$. Hence

$$\angle QSR = 180° - 30° = 150°.$$

Since QS is parallel to CR and CQ is not parallel to RS, $CQSR$ is a trapezium.

Solution (2): RS produced to meet BC must be a perpendicular to BC.

$$QSR = 90° + \angle BTS = 90° + 60° = 150°.$$

Solution (3): In quadrilateral $SRCQ$

$$\angle QSR + 30° + 60° + 35° + 60° + 25° = 360°$$

Hence $\angle QSR + 210° = 360°$ and $\angle QSR = 150°$.

5. a) Two numbers are respectively 20% and 50% more than a third number. What percentage is the first of the second?

   b) Three vessels of sizes 3 litres, 4 litres and 5 liters contain mixture of water and milk in the ratio 2:3, 3:7 and 4:11 respectively. The contents of all the three vessels are poured into a single vessel. What is the ratio of water to milk in the resultant mixture?

   Solution:

   (a) Let a, b be the two numbers and the third number be X. Given that $a = X + \frac{20}{100}X = \frac{6}{5}X$ and $b = X + \frac{50}{100}X = \frac{3}{2}X$. Dividing the above two equations, we get $\frac{a}{b} = \frac{4}{5}$. So the first is 80% of the second.

   (b) The actual volume of water and milk in the 3 vessels are:

	Water	Milk
1st Vessel	$\frac{6}{5}$	$\frac{9}{5}$
2nd Vessel	$\frac{12}{10}$	$\frac{28}{10}$
3rd Vessel	$\frac{20}{15}$	$\frac{55}{15}$

   Thus the ratio of the water to milk when the contents of the vessels are poured into a single vessel is

$$\frac{\frac{6}{5} + \frac{12}{10} + \frac{20}{15}}{\frac{9}{5} + \frac{28}{10} + \frac{55}{15}} = \frac{\frac{56}{15}}{\frac{124}{15}} = \frac{56}{124} = \frac{14}{31}$$

   The ratio of water to milk in the resultant mixture is $14 : 31$.

6. It is a well-known fact that Mahatma Gandhi was the man responsible for getting us the freedom. We got independence in 1947. Mahatma was born in 1869. Find the smallest numbers by which

   a) 1869 should be multiplied to get a product which ends in 1947

b) 1947 should be multiplied to get a product which ends in 1869.

(The method you use to obtain the required numbers should also be given).

Solution: Since we are interested in only the last four digits of the product $1869 \times N$ for a number N, it is sufficient to consider four digit numbers for N. Let

$$N = 10^3a + 10^2b + 10c + d$$

where a, b, c, d are integers between 0 and 9. Note that the last digit of the product $1869 \times N$ depends only on the last digits of 1869 and N, last two digits of the product $1869 \times N$ depends only on the last two digits of 1869 and the last two digits of N and so on. Thus, we must have

$$9d = 7 \mod 10 \tag{1}$$

$$69 \times (10c + d) = 47 \mod 10^2 \tag{2}$$

$$869 \times (100b + 10c + d) = 947 \mod 10^3 \tag{3}$$

$$1869 \times (1000a + 100b + 10c + d) = 1947 \mod 10^4 \tag{4}$$

From (1), we get $d = 3$. Substituting this in (2), we get

$$69 \times (10c + 3) = 47 \mod 100 \tag{5}$$

This implies

$$690c + 207 = 47 \mod 100$$

or

$$90c + 7 = 47 \mod 100$$

and hence $9c = 4 \mod 10$ and $c = 6$.
Now from (3) we get

$$869 \times (100b + 63) = 947 \mod 1000$$

Simplifying, we get

$$900b + 747 = 947 \mod 1000$$

or $9b = 2 \mod 10$ and hence $b = 8$. Finally, from (4) we have

$$1869 \times (1000a + 863) = 1947 \mod 10000$$

or $9000a + 2947 = 1947 \mod 10000$ and hence $9a = 9 \mod 10$. Thus $a = 1$ and the required number is 1863. For $N = 1863$, we have

$$1869 \times 1863 = 3481947$$

Since we have unique choices at each step above, it follows that 1863 is the only number less than 10000 with the required property and hence it is also the least number with the property.

A similar method can be used to show that 1927 is the least number such that 1947×1927 ends with the digits 1869. ($1947 \times 1927 = 3751869$).

Remark More generally, if k and l are any two numbers such that neither 2 nor 5 (prime factors of 10000) divide m or n, then we have a unique $N < 10^4$ such that the last digits of $k \times N$ is l.

FINAL – KAPREKAR CONTEST
NMTC at JUNIOR LEVEL
VII & VIII Standards

1. Prove the algebraic identity

$$a^3 - b^3 = \left\{\frac{a(a^3 - 2b^3)}{a^3 + b^3}\right\}^3 + \left\{\frac{b(2a^3 - b^3)}{a^3 + b^3}\right\}^3.$$

Solution:

$$\text{RHS} = \left\{\frac{a(a^3 + b^3 - 3b^3)}{a^3 + b^3}\right\}^3 - \left\{\frac{b(a^3 + b^3 - 3a^3)}{a^3 + b^3}\right\}^3.$$

Let $a^3 + b^3 = t$. Then RHS

$$\begin{aligned}
&= \frac{(a(t - 3b^3))^3 - (b(t - 3a^3))^3}{t^3} \\
&= \frac{a^3(t^3 - 9t^2b^3 + 27tb^6 - 27b^9)}{t^3} \\
&\quad - \frac{b^3(t^3 - 9t^2a^3 + 27ta^6 - 27a^9)}{t^3} \\
&= \frac{(a^3 - b^3)t^3 + 27ta^3b^3(b^3 - a^3) - 27a^3b^3(b^6 - a^6)}{t^3} \\
&= \frac{(a^3 - b^3)t^3}{t^3} \\
&= a^3 - b^3 = \text{LHS}.
\end{aligned}$$

2. The ratio between a two-digit number and the sum of the digits of that number is $a : b$. If the digits in the units place is n more than the digit in the tens place, prove that the number is given by $\dfrac{9na}{11b - 2a}$.

Solution: Let the number be $10x + y$. Given that

$$\frac{10x + y}{x + y} = \frac{a}{b} \tag{1}$$

$$y = x + n \tag{2}$$

Substituting (2) in (1) we get

$$\frac{11x+n}{2x+n} = \frac{a}{b} \tag{3}$$

Subtracting 1 from both sides of (3), we get

$$\frac{9x}{2x+n} = \frac{a-b}{b} \tag{4}$$

Cross multiplying and simplifying, we obtain

$$x = \frac{n(a-b)}{11b-2a} \tag{5}$$

Thus the number is

$$10x+y = 11x+n = \frac{11n(a-b)}{11b-2a} + n = \frac{9na}{11b-2a}$$

3. If $\dfrac{(a-b)(c-d)}{(b-c)(d-a)} = \dfrac{2012}{2013}$ find the value of $\dfrac{(a-c)(b-d)}{(a-b)(c-d)}$.

Solution: Given

$$\frac{(a-b)(c-d)}{(b-c)(d-a)} = \frac{2012}{2013} \tag{1}$$

Subtracting 1 from both sides, we get

$$\begin{aligned} \frac{(a-b)(c-d)}{(b-c)(d-a)} - 1 &= \frac{(ac-bc-ad+bd)-(bd-cd-ab+ac)}{(b-c)(d-a)} \\ &= \frac{cd+ab-bc-ad}{(b-c)(d-a)} \\ &= \frac{(b-d)(a-c)}{(b-c)(d-a)} \end{aligned} \tag{2}$$

and

$$\frac{2012}{2013} - 1 = -\frac{1}{2013} \tag{3}$$

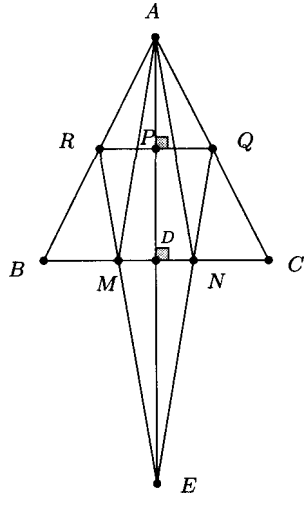
From (2) and (3) we get

$$\frac{(b-d)(a-c)}{(b-c)(d-a)} = -\frac{1}{2013} \tag{4}$$

Dividing (4) by (1) we get

$$\frac{(a-c)(b-d)}{(a-b)(c-d)} = -\frac{1}{2012}$$

4. Q, R are the midpoints of the sides AC, AB of the isosceles triangle ABC in which $AB = AC$. The median AD is produced to E so that $DE = AD$. EQ and ER are joined to cut BC in N and M respectively. Show that $AMEN$ is a rhombus.

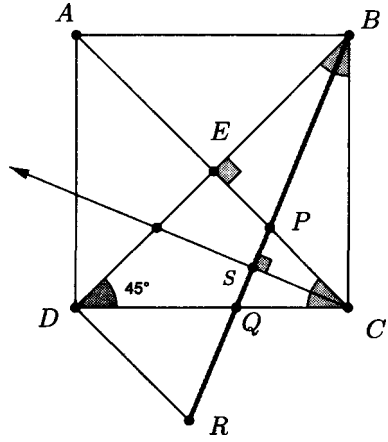


Solution: We will prove that all the four sides of $AMEN$ are equal.
Since $AB = AC$, AD is perpendicular to BC. Since QR is parallel to BC, AD is also perpendicular to QR. Clearly, P, the point of intersection of QR and AD is the mid point of QR and D is the mid point of BC.

Hence if we fold the triangle ABC along AE, Q will fall on R and C will fall on B. Hence the line QE will fall on ER and consequently, N will fall on M. Thus AN will fall on AM and hence $AM = AN$ and $EN = EM$. Now if we fold along the line BC, we deduce similarly that $AM = ME$ and $AN = NE$. Thus all the four sides of $AMEN$ are equal and it is a rhombus.

5. $ABCD$ is a square. The diagonals AC, BD, cut at E. From B a perpendicular is drawn to the bisector of $\angle DCA$ and it cuts AC at P and DC at Q. Prove that $DQ = 2PE$.

Solution: Extend BQ and draw a line through B parallel to EP to meet it at R. Let the bisector of PCD cut the line BQR at S. Clearly E is the midpoint of BD and EP is



parallel to DR. Hence P is the mid point of BR. By the mid point theorem, $DR = 2EP$.
$\angle DQR = \angle SQC = 90° - 22\frac{1}{2}° = 67\frac{1}{2}°$. Note that $\angle DBR$ is the angle between DB and DR is equal to the angle between the line EC perpendicular to DB and the line SC perpendicular to DR and thus equal to $22\frac{1}{2}°$. Now, $\angle QRD = 90° - \angle DBR = 67\frac{1}{2}°$ and hence $\angle DQR = \angle QRD$ and $DQ = DR$. Since $DR = 2EP$, it follows that $DQ = 2EP$.

6. a) A two digit number is equal to six times the sum of its digits.

Prove that the two digit number formed by interchanging the digits is equal to five times the sum of its digits.

Solution: Let $10a + b$ be the two digit number. Given that $10a + b = 6(a + b)$. Thus $4a = 5b$. The number obtained by interchanging the digits of the given number is $10b + a = 5b + 5b + a = 5b + 4a + a = 5(a + b)$.

b) Show that $\dfrac{10^{2013} + 1}{10^{2014} + 1} > \dfrac{10^{3013} + 1}{10^{3014} + 1}$

Solution: Let us put $a = 10^{2013}$ and $b = 10^{1000}$. Then we need to prove that

$$\frac{a+1}{10a+1} > \frac{ab+1}{10ab+1}$$

This is equivalent to showing

$$(a+1)(10ab+1) > (ab+1)(10a+1)$$

This holds if and only if

$$10a^2b + 10ab + a + 1 > 10a^2b + ab + 10a + 1$$

or we must prove $9ab > 9a$ or $b > 1$. Since $b = 10^{1000}$, it is obvious that $b > 1$ and the required inequality holds.

7. a) For any two natural numbers m, n prove that

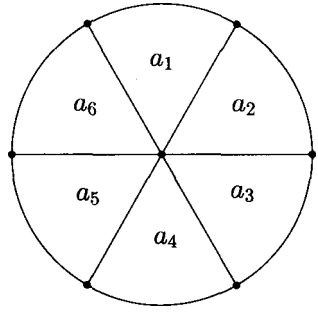
$(m^3 + n^3 + 4)$ cannot be a perfect cube.

Solution: Any integer N can leave a remainder 0, 1 or 2 when divided by 3 and hence can be written in the form $3k$ or $3k + 1$ or $3k + 2$ for some integer k. When the number is cubed, then we have $N^3 = 27k^3$ or $N^3 = 27k^3 + 27k^2 + 9k + 1$ or $N^3 = 27k^3 + 54k^2 + 36k + 8$. Thus when N^3 is divided by 9, the remainders can only be $0, 1$ or 8. When m, n take different values, we compute the remainder when $m^3 + n^3 + 4$ is divided by 9. This is given in the following table:

	$m = 3k$	$m = 3k + 1$	$m = 3k + 2$
$n = 3l$	4	5	3
$n = 3l + 1$	5	6	4
$n = 3l + 2$	3	4	2

Since none of the entries are $0, 1$ or 8, it follows that $m^3 + n^3 + 4$ can not be a perfect cube.

b) A circle is divided into six sectors and the six numbers 1, 0, 1, 0, 0, 0 are written clockwise, one in each sector. One can add 1 to the numbers in any two adjacent sectors. Is it possible to make all the numbers equal? If so after how many operations can this be achieved?



Solution: Taking the six numbers as a_1, a_2, a_3, a_4, a_5 and a_6, we find that $a_1 - a_2 + a_3 - a_4 + a_5 - a_6 = 1 - 0 + 1 - 0 + 0 - 0 = 2$. If 1 is added to any two adjacent sectors then the sum $a_1 - a_2 + a_3 - a_4 + a_5 - a_6$ is not altered.

After any number of times if the numbers in the sectors are the same then $a_1 - a_2 + a_3 - a_4 + a_5 - a_6 = 0$ but we started with 2. Hence it is not possible to make the numbers in all sectors the same.

We can not get all the numbers equal even after any number of operations.

8. a) All natural numbers from 1 to 2013 are written in a row in that order. Can you insert $+$ and $-$ signs between them so that the value of the resulting expression is zero? If it is possible, how many $+$ and $-$ signs should be inserted? Justify

your answer by giving clear reasoning.

Solution: We have $1 + 2 + 3 + \cdots + 2013 = \frac{2013 \times 2014}{2} = 2013 \times 1007$, odd number.

Changing a plus $(+)$ to minus $(-)$ does not change the parity of the expression.

Thus any number, we get by writing the number 1 to 2013 in a row and placing plus and minus signs between them must be odd. So the result cannot be zero.

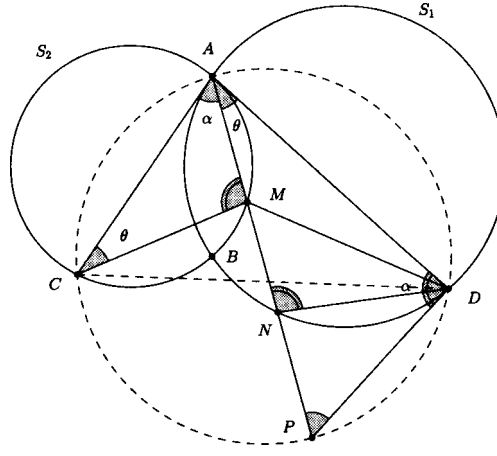
b) The natural numbers $1, 2, 3, \cdots$ are partitioned into subsets $S_1 = \{1\}$, $S_2 = \{2, 3\}$, $S_3 = \{4, 5, 6\}$, $S_4 = \{7, 8, 9, 10\}$ and so on. What are the greatest and least numbers in the set S_{2013}?

Solution: The set S_i contains i elements. Hence $\bigcup_{i=1}^{k} S_i$ contains $1 + 2 + \cdots + k = \frac{k(k+1)}{2}$ elements. Hence the last number in S_{2013} is $\frac{2013 \times 2014}{2} = 2027091$. Since S_{2013} contains 2013 numbers, the first number in S_{2013} is $2027091 - 2013 + 1 = 2025078$.

FINAL – BHASKARA CONTEST
NMTC at JUNIOR LEVEL
IX & X Standards

1. Two circles S_1 and S_2 intersect at points A and B. The tangent at A to S_1 meets S_2 at C and the tangent at A to S_2 meets S_1 at D. A line through A interior to the angle CAD meets S_1 at M and S_2 at N and meets the circumcircle of triangle ACD at P. Prove that $AM = NP$.

Solution:



Let $\angle ACM = \theta$. Then, using alternate segment theorem, $\angle MAD = \theta$. Similarly, $\angle CAM = \angle ADN = \alpha$. Thus $\triangle MAC \sim \triangle NDA$ and

$$\frac{AM}{AC} = \frac{DN}{AD} \tag{1}$$

Clearly, $ACPD$ is a cyclic quadrilateral. Since angles in the same segment are equal, we have $\angle ACD = \angle APD$ and $\angle PDC = \angle PAC = \alpha = \angle ADN$.

Now,

$$\angle PDC - \angle NDC = \angle ADN - \angle NDC$$

and hence $\angle PDN = \angle CDA$. Thus $\triangle NPD \sim \triangle ACD$ and

$$\frac{NP}{AC} = \frac{DN}{AD} \tag{2}$$

From (1) and (2) we get

$$\frac{AM}{AC} = \frac{NP}{AC} \Rightarrow AM = NP.$$

2. Let a, b, c, d be positive real numbers. Show that

$$\frac{ab+bc+ca}{a^3+b^3+c^3} + \frac{ab+bd+da}{a^3+b^3+d^3} + \frac{ac+cd+da}{a^3+c^3+d^3} + \frac{bc+cd+db}{b^3+c^3+d^3}$$

$$\leq \min\left\{\frac{a^2+b^2}{(ab)^{3/2}} + \frac{c^2+d^2}{(cd)^{3/2}}, \frac{a^2+c^2}{(ac)^{3/2}} + \frac{b^2+d^2}{(bd)^{3/2}}, \frac{a^2+d^2}{(ad)^{3/2}} + \frac{b^2+c^2}{(bc)^{3/2}}\right\}.$$

Solution: Even though the problem looks very difficult, the solution involves only some basic ideas of inequalities.
From AM-GM inequality applied to the positive reals a^3, b^3, c^3 we get

$$a^3 + b^3 + c^3 \geq 3abc \tag{1}$$

Now,

$$\begin{aligned} \frac{ab+bc+ca}{a^3+b^3+c^3} &= \frac{abc\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)}{a^3+b^3+c^3} \\ &\leq \frac{1}{3}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right), \text{using (1)} \end{aligned} \tag{2}$$

Hence

$$\frac{ab+bc+ca}{a^3+b^3+c^3} + \frac{ab+bd+da}{a^3+b^3+d^3} + \frac{ac+cd+da}{a^3+c^3+d^3} + \frac{bc+cd+db}{b^3+c^3+d^3}$$

$$\leq \frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}$$

From AM-GM inequality applied to the positive reals a^2, b^2 we get

$$a^2 + b^2 \geq 2ab \tag{3}$$

Adding $a^2 + b^2$ to both sides of (3), we get

$$2(a^2 + b^2) \geq (a + b)^2$$

and hence

$$\sqrt{2(a^2 + b^2)} \geq a + b \tag{4}$$

Also from (3) we get

$$\sqrt{(a^2 + b^2)} \geq \sqrt{2ab} \tag{5}$$

$$a^2 + b^2 \geq \sqrt{ab}(a + b) \tag{6}$$

$$\frac{1}{a} + \frac{1}{b} = \frac{a+b}{ab} \leq \frac{a^2 + b^2}{(ab)^{\frac{3}{2}}} \tag{7}$$

Thus

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} \leq \frac{a^2 + b^2}{(ab)^{\frac{3}{2}}} + \frac{c^2 + d^2}{(cd)^{\frac{3}{2}}} \tag{8}$$

Similarly,

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} \leq \frac{a^2 + c^2}{(ac)^{\frac{3}{2}}} + \frac{b^2 + d^2}{(bd)^{\frac{3}{2}}} \tag{9}$$

and

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} \leq \frac{a^2 + d^2}{(ad)^{\frac{3}{2}}} + \frac{b^2 + c^2}{(bc)^{\frac{3}{2}}} \tag{10}$$

Finally,

$$\begin{aligned}
&\frac{ab + bc + ca}{a^3 + b^3 + c^3} + \frac{ab + bd + da}{a^3 + b^3 + d^3} + \frac{ac + cd + da}{a^3 + c^3 + d^3} + \frac{bc + cd + db}{b^3 + c^3 + d^3} \\
&\leq \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} \\
&\leq \min\left\{ \frac{a^2 + b^2}{(ab)^{3/2}} + \frac{c^2 + d^2}{(cd)^{3/2}}, \frac{a^2 + c^2}{(ac)^{3/2}} + \frac{b^2 + d^2}{(bd)^{3/2}}, \frac{a^2 + d^2}{(ad)^{3/2}} + \frac{b^2 + c^2}{(bc)^{3/2}} \right\}
\end{aligned}$$

3. Find prime numbers p such that $4p^2+1$ and $6p^2+1$ are also prime numbers.

   Solution: Let us check for the first few primes:

p	$4p^2+1$	$6p^2+1$
2	17	25
3	37	55
5	101	151

   For $p=5$, both $4p^2+1$ and $6p^2+1$ are primes.
   Let p be a prime greater then 5. By Fermat's theorem, $p^4 \equiv 1 \mod 5$. Now,

$$(4p^2+1)(6p^2+1) = 24p^4+10p^2+1 \equiv 0 \mod 5$$

   Thus 5 must divide one of $4p^2+1$ or $6p^2+1$ and hence both can not be primes. Thus $4p^2+1$ and $6p^2+1$ are both primes only when $p=5$.

4. a) Find all positive integral solutions x, y, z of the equation $xy+yz+zx = xyz+2$.

   Solution: By symmetry, it can be assumed that $x \le y \le z$.

   Dividing by xyz we get,

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z} = 1+\frac{2}{xyz}$$

   Hence

$$1 < 1+\frac{2}{xyz} = \frac{1}{x}+\frac{1}{y}+\frac{1}{z} \le \frac{3}{x}$$

   Thus $x<3$ and since x is a positive integer, we have $x=1$ or $x=2$.
   If $x=1$ then the equation becomes $y+yz+z = yz+2$ or $y+z=2$. Again, since y, z are positive integers, we must have

$y = z = 1$.

If $x = 2$,

$$\frac{1}{y} + \frac{1}{z} = \frac{1}{2} + \frac{1}{yz}$$

Now,

$$\frac{1}{2} < \frac{1}{2} + \frac{1}{yz} = \frac{1}{y} + \frac{1}{z} \leq \frac{2}{y} \quad \text{since } y \leq z$$

Thus $y < 4$.

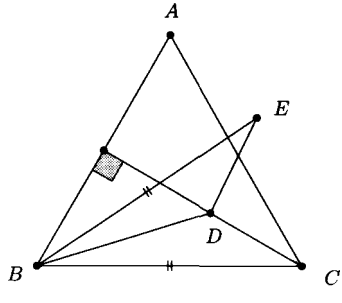
A simple check yields $y = 3$ and $z = 4$. The solutions are

$$(x, y, z) \equiv (1,1,1), (2,3,4), (2,4,3), (3,2,4), (3,4,2), (4,2,3), (4,3,2)$$

b) ABC is an equilateral triangle. D is a point inside the triangle such that $DA = DB$. E is a point that satisfies the two conditions

i) $\angle DBE = \angle DBC$ and ii) $BE = AB$.

Find the measure of the $\angle DEB$.



Solution: D lies on the perpendicular bisector of AB. Hence $\angle BCD = 30°$ and the triangles BDE and BDC are congruent (BD is common, $\angle DBE = \angle DBC$ and $BE = BC$). Hence $\angle DEB = \angle DCB = 30°$.

5. a) Show that the numbers 1 to 15 cannot be divided into a group A of 2 numbers and a group B of 13 numbers in such a way that the sum of the numbers in B is equal to the product of the numbers in A.

Solution: Let us assume that there are 2 distinct numbers a, b

among 1 to 15 such that the given condition holds. Then

$$(1+2+3+4+\cdots+15)-(a+b) \;=\; ab$$

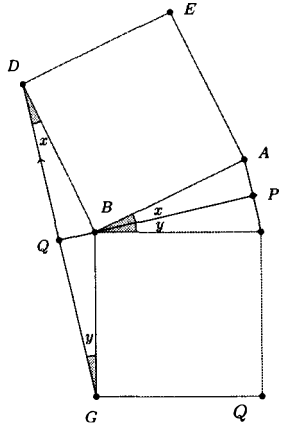
$$\Rightarrow \qquad a+b+ab=120$$

$$\Rightarrow \qquad 1+a+b+ab=121$$

$$\Rightarrow \qquad (1+a)(1+b)=11^2$$

This is not possible since $a \neq b$. Hence such a partition does not exist.

b) Squares $ABCD$ and $BCFG$ are drawn outside of a triangle ABC. Prove that if DG is parallel to AC then the triangle ABC is isosceles.



Solution: Since angle between two lines is equal to the angle between their perpendiculars, we have $\angle ABP = \angle BDQ$. Also, $\angle APB = \angle BQD = 90°$ and since $ABCD$ is a square, $AB = BD$. It follows that the triangles ABP and BD are congruent and

$$AP = BQ \tag{1}$$

Again, $\angle PBC = \angle BGQ$ (using angle between lines PB and PC is equal to the angle between their perpendiculars GQ and GB) and $BC = BG$ (sides of the square $BCFG$). Hence triangles BPC and GBQ are congruent and

$$PC = BQ \tag{2}$$

From (1) and (2) we get $AP = PC$ and consequently the triangle ABC is isosceles with $AB = BC$.

6. There are 13 white, 15 black and 17 red beads on a table. You have many number of beads of these colours with you. In one step 2 beads on the table of different colours are chosen by you and you replace each one by a bead of the third colour from you. After how many such steps you will have all the beads of the same colour?

Solution: If we start with w white beads, b black beads and r red beads, after one step, we end up with one of the following: $(w+2, b-1, r-1)$ or $(w-1, b+2, r-1)$ or $(w-1, b-1, r+2)$ beads of respective colors. In all these cases we observe that the difference between the number of white beads and the black beads is either $w-b+3$ or $w-b-3$ or $w-b$. In all the cases, the remainder when the difference is divided by 3 remains the same. When we start with 13 white beads and 15 black beads, the difference is 2 modulo 3 and hence is invariant after any number of steps. Thus when all the beads have the same color, then we must have either $(0, 0, 45)$ or $(0, 45, 0)$ or $(45, 0, 0)$. In none of these configurations is the difference between white beads and black beads is 2 modulo 3. Thus we can not end up with all beads of the same color.

7. a) If a, b, c, d are positive real numbers such that $a+b+c+d = 1$, show that $\dfrac{a^3}{b+c} + \dfrac{b^3}{c+d} + \dfrac{c^3}{d+a} + \dfrac{d^3}{a+b} \geq \dfrac{1}{8}$.

Solution: Applying AM-GM inequality to $\dfrac{a^3}{b+c}, \dfrac{b+c}{16}, \dfrac{1}{32}$, we get

$$\frac{a^3}{b+c} + \frac{b+c}{16} + \frac{1}{32} \geq 3\left(\frac{a^3(b+c)}{(b+c)16 \times 32}\right) = \frac{3a}{8}$$

Similarly,

$$\frac{b^3}{c+d}+\frac{c+d}{16}+\frac{1}{32}\geq\frac{3b}{8}$$
$$\frac{c^3}{d+a}+\frac{d+a}{16}+\frac{1}{32}\geq\frac{3c}{8}$$
$$\frac{d^3}{a+b}+\frac{a+b}{16}+\frac{1}{32}\geq\frac{3d}{8}$$

Adding the above, we get

$$\frac{a^3}{b+c}+\frac{b^3}{c+d}+\frac{c^3}{d+a}+\frac{d^3}{a+b}+\frac{a+b+c+d}{8}+\frac{1}{8}\geq\frac{3}{8}(a+b+c+d)$$

Since $a+b+c+d=1$ the inequality follows.

b) A 4-digit number n not containing the digit 9 is a square of an integer. If we increase every digit of n by 1 we get a square of another integer again. Find all such n.

Solution: Let $n=a^2$ given $n+1111=b^2$.

$$b^2-a^2=(b-a)(b+a)=1111=101\times 11$$

Since 11 and 101 are primes, either $b-a=11$ and $b+a=101$ or $b+a=1111$ and $b-a=1$. In the first case, we get $a=45$ and $n=2025$. When we increase every digit of n by 1, we get $3136=56^2$.

In the second case, we get $a=555$ but then $n=a^2$ is not a four digit number. Hence the only number with the property is 2025.

8. a) Find all positive real numbers x,y,z which satisfy the following equations simultaneously.

$$x^3+y^3+z^3 = x+y+z \quad (1)$$
$$x^2+y^2+z^2 = xyz \quad (2)$$

Solution:

$$xyz=x^2+y^2+z^2>y^2+z^2\geq 2yz$$

Hence $x > 2$. Similarly, we deduce that $y > 2$ and $z > 2$. But then $x^3 > x$, $y^3 > y$ and $z^3 > z$. Now,

$$x^3 + y^3 + z^3 > x + y + z$$

and (1) can not hold.
Thus there are no such numbers.

b) Do there exist 10 distinct integers such that the sum of any 9 of them is a perfect square?

Solution: Let $a_1, a_2, \cdots, a_{10}$ be distinct integers satisfying the given condition.

Let S be their sum. Then for each i, $1 \leq i \leq 10$, we have $S - a_i = K_i^2$ for some integer K_i. Let $T = K_1^2 + K_2^2 + \cdots + K_{10}^2$. Now

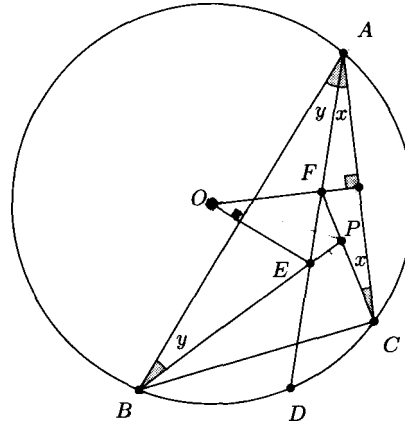
$$\sum_{i=1}^{10} (S - a_i) = 9S = T$$

We have $a_i = S - (S - a_i) = \frac{T}{9} - K_i^2$. Hence if we choose distinct integers $K_1, K_2, \ldots, K_{10}$ such that the sum of their squares is a multiple of 9, we can find the a_i. For example, we can take $K_i = 3i$ for $i = 1, 2, \ldots, 10$ and we get $376, 349, 304, 241, 160, 61, -56, -191, -344, -515$ as one set of possible integers satisfying the given condition.

FINAL – RAMANUJAN CONTEST
NMTC at INTER LEVEL
XI & XII Standards

1. ABC is a triangle in which $AB > AC > BC$. D is a point on the minor arc BC of the circumcircle of the triangle ABC. O is the circumcentre. E and F are the intersection points of the line AD with the perpendiculars from O to AB and AC respectively. P is the point of intersection of BE and CF. If $PB = PC + PO$, find the angle A of the triangle ABC.

Solution: Since F is on the perpendicular bisector of AC,



$AF = FC$. Also,

$$\angle BPC = \angle AEP + \angle CFD = 2(\angle BAD + \angle CAD) = \angle BOC$$

Thus B, C, P, O are concyclic. By Ptolemy's Theorem, $PB \cdot OC = PC \cdot OB + PO \cdot BC$ and hence $(PB - PC) \cdot OC = PO \cdot BC$. But given that $PB - PC = PO$ and hence

$OC = BC$. Thus triangle OBC is equilateral and $\angle BAC = \frac{1}{2}\angle BOC = 30°$. The same proof holds good even if O lies inside the triangle ABC.

2. a) For the positive integer n define

$$f(n) = 1^n + 2^{n-1} + 3^{n-2} + \cdots + (n-2)^3 + (n-1)^2 + n^1.$$

What is the minimum value of $\dfrac{f(n+1)}{f(n)}$?

b) ABC is an isosceles triangle in which $AB = AC$. The bisector of $\angle B$ meets AC at D and it is given that $BC = BD + AD$. Find $\angle A$ of the triangle ABC.

Solution a): Let $g(n) = \dfrac{f(n+1)}{f(n)}$. We need the minimum value of g. Clearly,

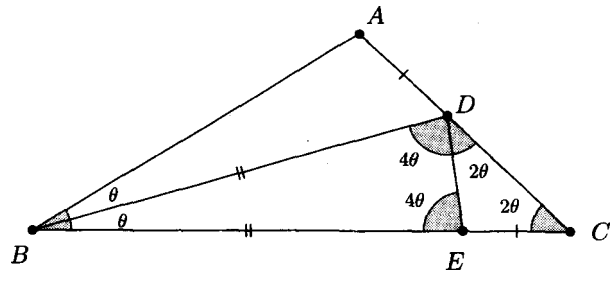
$$g(1) = 3, \quad g(2) = \frac{8}{3}, \quad g(3) = \frac{22}{8}$$

$$g(4) = \frac{65}{22}, \quad g(5) = \frac{209}{65}, \quad g(6) = \frac{732}{209}$$

Among the above, $g(2) = \dfrac{8}{3}$ is the least. We will estimate the value of $g(n)$ for $n > 6$.

$$\begin{aligned} f(n+1) &> 1^{n+1} + 2^n + \cdots + 6^{n-4} + \cdots + (n-1)^3 + n^2 \\ &> 1^{n+1} + 2^n + \cdots + 3(6^{n-5} + \cdots + (n-1)^2 + n) \\ &= 1^{n+1} + \cdots + 5^{n-3} + 3(f(n) - 1^n - \cdots - 5^{n-4}) \\ &= 3f(n) + 2(5^{n-4} - 1) + 2^{n-1}(2^{n-5} - 1) \\ &> 3f(n) \end{aligned}$$

Hence $g(n) = \dfrac{f(n+1)}{f(n)} > 3$ for $n > 6$. Thus the minimum value of $g(n)$ is $\dfrac{8}{3}$.



Solution b): Let E be a point BC such that $BD = BE$. From the given condition we have

$$EC = AD \tag{1}$$

By bisector theorem we have

$$\frac{AD}{DC} = \frac{AB}{BC} \tag{2}$$

Consider the triangles $\triangle CED$ and $\triangle CAB$. $\angle C$ is common and from (1) and (2) we get

$$\frac{EC}{DC} = \frac{AB}{BC}$$

Hence $\triangle CED$ is similar to $\triangle CAB$. If $\angle ABD = \theta$, then it is easy to see that $\angle BDE = \angle BED = 4\theta$ and it follows that $\theta = 20°$ and $\angle A = 180° - 4\theta = 100°$.

3. Let n be a positive integer and S_n be the set of all positive integer divisors of n (including 1 and itself). Prove that at most half of the elements of S_n have their units digit equal to 3.

Solution: Let us consider the following 3 cases.

(a) Suppose that n is a multiple of 5, say, $n = 5K$. Let $d_1, d_2, \cdots, d_m$ be the elements of S_n with their last digit 3. Note that all these must be divisors of K. Hence $5d_1, 5d_2, \cdots, 5d_m$ are distinct and also elements of S_n. Thus the assertion is valid in this case.

(b) Suppose that n is not a multiple of 5. Suppose also that the last digit of every prime divisor of n is either 1 or 9. Then the last digit of any element in S_n is either 1 or 9. Thus in this case also the statement is true.

(c) Suppose now that n is not a multiple of 5 and there exists a prime divisor of p in S_n such that the last digit of p is either 3 or 7.

Let $n = p^r q$ where p, q are positive integers and p and q are prime to each other. Let $S_q = \{a_1, a_2, \cdots, a_k\}$ be the set of all positive divisors of q.

We write the elements of S_n as

$$a_1, a_1p, a_1p^2, \ldots, a_1p^r$$
$$a_2, a_2p, a_2p^2, \ldots, a_2p^r$$
$$\cdots\cdots$$
$$a_k, a_kp, a_kp^2, \ldots, a_kp^r$$

For any $d_i = a_sp^l \in S_n$, let us choose

$$e_i = \begin{cases} a_sp^{l+1} & l < r \\ a_sp^{l-1} & l = r \end{cases}$$

Then $e_i \in S_n$.
If the last digit of d_i is 3, then the last digit of e_i is not 3, since the last digit of p is either 3 or 7.

Let $d_i, d_j \in S_n$ be such that $d_i \neq d_j$ and both their last digits are 3. Then the corresponding e_i and e_j are also

different. For, if $e_i = e_j = a_s p^l$ then we may assume that $\{d_i, d_j\} = \{a_s p^{l-1}, a_s p^{l+1}\}$ and hence $d_i = d_j p^2$. But as the last digit of p is 3 or 7, the last digit of p^2 is always 9. This implies d_i and d_j can not have the same last digit and hence $d_i \neq d_j p^2$. Thus e_i and e_j are different.
Hence for every $d_i \in S_n$, with its last digit equal to 3 there is a corresponding $e_i \in S_n$ with its last digit not equal to 3 and different d_i s have different e_i.

Consequently, it follows that at most half the elements of S_n can have their last digits equal to 3.

4. a) Let A be a set of 8 elements. Find the maximum number of 3 - element subsets of A, such that the intersection of any two of them is not a 2 element set.
b) a, b, c, d, are all positive reals and

$$\frac{1}{1+a^4} + \frac{1}{1+b^4} + \frac{1}{1+c^4} + \frac{1}{1+d^4} = 1.$$

Prove that $abcd \geq 3$.

Solution a): Let the number of elements of a set S be denoted by $|S|$.

Let $B_1, B_2, \cdots, B_n \subseteq A$, such that $|B_i| = 3, |B_i \cap B_j| \neq 2$ for $i, j = 1, \cdots, n$.

If $a \in A$ belongs to $B_1, B_2, \cdots, B_k$ then $|B_i \cap B_j| = 1$ for $i = 1, \cdots, k$.

Since $8 = |A| \geq |B_1 \cup B_2 \cdots \cup B_k| = 1 + 2k$, we get $k \leq 3$.

From this we see that every element of A is in at most 3 B_i's and $3n \leq 8 \times 3 \Rightarrow n \leq 8$.

To show 8 is possible, just consider

$B_1 = \{1,2,3\}$, $B_2 = \{1,4,5\}$, $B_3 = \{1,6,7\}$, $B_4 = \{8,3,4\}$,

$B_5 = \{8,2,6\}$, $B_6 = \{8,5,7\}$, $B_7 = \{3,5,6\}$, $B_8 = \{2,4,7\}$.

Solution b): Put

$$x = \frac{1}{1+a^4}, \quad y = \frac{1}{1+b^4}, \quad z = \frac{1}{1+c^4}, \quad t = \frac{1}{1+d^4}$$

Then given that $x + y + z + t = 1$. Also

$$a^4 = \frac{1-x}{x}, \quad b^4 = \frac{1-y}{y}, \quad c^4 = \frac{1-z}{z}, \quad d^4 = \frac{1-t}{t}$$

and we need to prove

$$a^4b^4c^4d^4 = \frac{1-x}{x} \cdot \frac{1-y}{y} \cdot \frac{1-z}{z} \cdot \frac{1-t}{t} \geq 81.$$

Now,

$$\begin{aligned} &\frac{1-x}{x} \cdot \frac{1-y}{y} \cdot \frac{1-z}{z} \cdot \frac{1-t}{t} \\ &= \frac{y+z+t}{x} \cdot \frac{x+z+t}{y} \cdot \frac{x+y+t}{z} \cdot \frac{x+y+z}{t} \\ &\geq \frac{3(yzt)^{\frac{1}{3}}}{x} \cdot \frac{3(xzt)^{\frac{1}{3}}}{y} \cdot \frac{3(xyt)^{\frac{1}{3}}}{z} \cdot \frac{3(xyz)^{\frac{1}{3}}}{t} = 81 \end{aligned}$$

completing the proof.

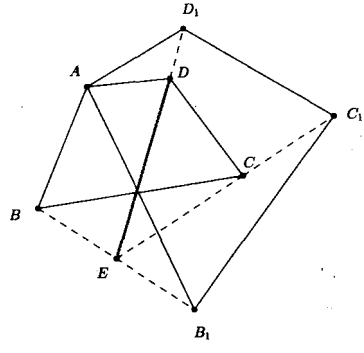
5. In a plane there are two similar, convex quadrilaterals $ABCD$ and $AB_1C_1D_1$ such that C, D are inside $AB_1C_1D_1$ and B is outside $AB_1C_1D_1$. Prove that if the lines BB_1, CC_1, DD_1 are concurrent, then $ABCD$ is cyclic. Is the converse true?

Solution: Since the convex quadrilaterals $ABCD$ and $AB_1C_1D_1$ are similar, we have

$$\frac{AB}{AB_1} = \frac{AC}{AC_1} = \frac{AD}{AD_1} \tag{1}$$

Clearly, the triangles ABC and AB_1C_1 are similar. Hence $\angle BAC = \angle B_1AC_1$. Now,

$$\angle BAB_1 = \angle BAC - \angle B_1AC = \angle B_1AC_1 - \angle B_1AC = \angle CAC_1$$



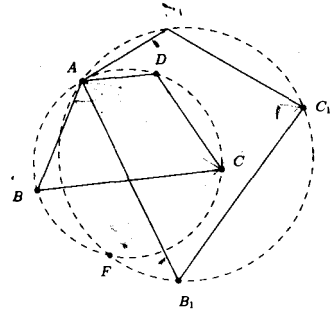
Similarly, $\angle CAC_1 = \angle DAD_1$. By (1) and the above equalities we deduce that the triangles BAB_1, CAC_1 and DAD_1 are similar to each other. Hence

$$\angle AB_1B = \angle AC_1C = \angle AD_1D \tag{2}$$

If BB_1, CC_1, DD_1 are concurrent at E, then (2) can be restated as

$$\angle AB_1E = \angle AC_1E = \angle AD_1E \tag{3}$$

Thus, A, B_1, C_1, D_1, E are concyclic and the quadrilateral



$AB_1C_1D_1$ is cyclic. By similarity, $ABCD$ is also cyclic.

Converse is also true. This can be seen as follows:

Let $ABCD$ be cyclic. Then $AB_1C_1D_1$ is also cyclic because of similarity. Let the two circumcircles intersect at A and F. Let O be the circumcentre of $ABCD$ and O_1, the circumcentre of $AB_1C_1D_1$. Since triangle AOD is similar to triangle AO_1D_1, $\angle AOD = \angle AO_1D_1$. Now,

$$\angle AFD = \frac{1}{2}\angle AOD = \frac{1}{2}\angle AO_1D = \angle AFD_1$$

Hence the line DD_1 passes through the point F. Similarly, the lines BB_1 and CC_1 also pass through F. Thus BB_1, CC_1 and DD_1 are concurrent.

6. Prove that if the integer n is not divisible by 5, then the polynomial $f(x) = x^5 - x + n$ cannot be factored as the product of two non-constant polynomials with integer coefficients.

Solution: Suppose $f(x)$ can be factorized as a product of polynomials with integer coefficients. Then it has a factor with degree 1 or 2. If one of the factors has degree 1, then we can write $f(x) = (x - a)g(x)$, where a is an integer. Putting $x = a$, we get $a^5 - a + n = 0$. But for any integer a, by Fermat's theorem, $a^5 - a \equiv 0 \mod 5$, it follows that $n \equiv 0 \mod 5$, a contradiction.

Now suppose that $f(x)$ has a quadratic polynomial as a factor. Let $f(x) = (x^2 + ax + b)g(x)$ where a, b are integers. It is easy to see that the remainder when we divide $x^5 - x + n$ by $x^2 + ax + b$ is $(a^4 - 3a^2b + b^2 - 1)z + (a^3b - 2a^2b + n)$. Since this remainder must be zero, we must have

$$\begin{aligned} a^4 - 3a^2b + b^2 - 1 &= 0 \\ a^3b - 2ab^2 + n &= 0 \end{aligned}$$

Now,

$$0 = a(a^4 - 3a^2b + b^2 - 1) + 3(a^3b - 2ab^2 + n) = a^5 - a - 5ab^2 + 3n$$

or $3n = -(a^5 - a) - 5ab^2$. Since $a^5 - a$ is a multiple of 5 it follows that $3n$ and hence n is a multiple of 5, a contradiction. Thus the given polynomial cannot be factored as the product of two non-constant polynomials with integer coefficients.

Remark More generally, for any prime p, the polynomial $X^p - X + k$ can not be factored as a product of polynomials with integer coefficients when k is not a multiple of p. It can also be shown that $X^{p^2} - X + k$ is also irreducible when k is not a multiple of p.

7. a) One may perform the following two operations on a positive integer.

(i) Multiply it by any positive integer.

(ii) Delete zeros in its decimal representation.

Prove that for every positive integer n, one can perform a sequence of these operations that will transform n into a single digit number.

b) Show that $1! + 2! + 3! + \cdots + 2013!$ can not be written as n^k for any integer n and integer $k \geq 2$.

Solution a): By Pigeon hole principle, at least two of $(n+1)$ numbers 1,11,111, $\cdots, \underbrace{111\cdots11}_{n+1 \text{ digits}}$ have the same remainder when divided by n.

So taking the difference of two of these numbers, we get a number of the form $11,\cdots 100\cdots 0$ which is a multiple of n.

Now, perform operation (i) on n to get such a multiple.

Then perform operation (ii) to delete the zeros (if any).

If the resulting number has more than one digit, then

(A) Multiply by 82 to get a number $911\cdots102$

(B) Delete the zero and multiply by 9 to get the number $8200\cdots08$

(C) Delete the zeros to get 828

(D) Multiply by 25: $828 \times 25 = 20700$

(E) Delete zeros to get 27

(F) Multiply by 4: $27 \times 4 = 108$

(G) Delete zero to get 18

(H) Multiply by 5: $18 \times 5 = 90$

(I) Delete zero to get 9, a single digit number.

Solution b): Let $S = 1! + 2! + 3! + \cdots + 2013!$. Clearly, $1! + 2! + 3! + 4! = 33$, is not a power of an integer.

For $k \geq 5$, $k! \equiv 0 \mod 10$ and hence

$$S \equiv (33 + 0) \mod 10 = 3 \mod 10$$

Since no square ends in 3, S can not be a square (and hence also can not be an even power of an integer).

For any $n \geq 9$, $n!$ is a multiple of 27 and $1!+2!+3!+4!+\cdots+8!$ is a multiple of 9 but not 27. Thus S is a multiple of 9 but not of 27. If $S = n^k$ for some n and $k \geq 3$, then n must be a multiple of 3 but then it would follow that S is a multiple of 27, a contradiction. Hence S can not be the power an integer.

8. a) $\lfloor x \rfloor$ denotes the floor function (the greatest integer function). Let r be a real number for which

$$\left\lfloor r + \frac{19}{100} \right\rfloor + \left\lfloor r + \frac{20}{100} \right\rfloor + \left\lfloor r + \frac{21}{100} \right\rfloor + \cdots + \left\lfloor r + \frac{91}{100} \right\rfloor = 546.$$

Solve the equation $x + \lfloor 100r \rfloor = 2013$.

b) For all distinct positive integers m and n prove $(2013)^{2^n} + 2^{2^n}$ is relatively prime to $(2013)^{2^m} + 2^{2^m}$.

Solution a): The number of terms in the equation given is $91 - 19 + 1 = 73$. Each term is equal to either $\lfloor r \rfloor$ or $\lfloor r \rfloor + 1$. Let k be the least positive integer such that

$$\left\lfloor r + \frac{k}{100} \right\rfloor = \lfloor r \rfloor + 1$$

We have

$$(k - 1 - 18)\lfloor r \rfloor + (91 - k + 1)(\lfloor r \rfloor + 1) = 546$$

Thus $73 \times \lfloor r \rfloor + (92 - k) = 546$. Since $73 \times 8 = 584$, $\lfloor r \rfloor < 8$. If $\lfloor r \rfloor = 7$, then $k = 57$ and $\lfloor r + \frac{56}{100} \rfloor = 7$ and $\lfloor r + \frac{57}{100} \rfloor = 8$. Hence $743 \leq 100r < 744$ and $\lfloor 100r \rfloor = 743$.

The given equation $x + [100r] = 2013$ can be written as $x + 743 = 2013$ and hence $x = 2013 - 743 = 1270$.

Solution b): Without loss of generality, we can assume $m > n$.

For any prime p dividing $2013^{2^n} + 2^{2^n}$

we have $(2013)^{2^n} \equiv -2^{2^n} \mod p$

Squaring the equation $(m - n)$ times, we obtain

$(2013)^{2^m} \equiv 2^{2^m} \mod p$.

Since 2013 is odd, we have $p \neq 2$.

$\therefore \quad 2^{2^m} + 2^{2^m} = 2^{2^m + 1} \not\equiv 0 \pmod{p}$.

$\Rightarrow \quad (2013)^{2^m} \equiv 2^{2^m} \not\equiv -2^{2^m} \pmod{p}$.

$\Rightarrow \quad p \nmid (2013)^{2^m} + 2^{2^m}$.

Hence the given numbers are prime to each other.

ARYABHATA CONTEST
SENIOR – 2013

1. Find all possible pairs of positive integers (m, n) such that $m^3 + 11^3 = n^3$.

   Solution: We will show that no such pairs exist.

   $$1331 = 11^3 = n^3 - m^3 = (n-m)(n^2 + nm + m^2)$$

   Either $n - m = 1$ or $n - m$ is divisible by 11. If $n - m = 1$, then

   $$11^3 = (m+1)^3 - m^3 = 3m^2 + 3m + 1$$

   and hence $1330 = 3m(m+1)$, a contradiction since 1330 is not a multiple of 3.

   It follows that $n \equiv m \bmod 11$. Since $n > m$, we have $n = m + 11\lambda$ for some positive integer λ. Thus

   $$m^3 + 11^3 = m^3 + 3 \cdot 11 \cdot \lambda m^2 + 3 \cdot 11^2 \cdot \lambda^2 m + \lambda^3 11^3$$

   and hence

   $$11^3 = 3 \cdot 11 \cdot \lambda m^2 + 3 \cdot 11^2 \cdot \lambda^2 m + \lambda^3 11^3$$

   This is impossible since $\lambda > 0$. Thus the given equation has no solutions.

2. Several schools took part in a tournament. Each player played one match against each player from a different school and did not play anyone from the same school. The total number of boys taking part differed from the total number of girls by 1. The total number of matches with both players of the same sex differed by at most one from the total number of matches

with players of opposite sex. What is the largest number of schools that could have sent an odd number of players to the tournament?

Solution: Let there be n schools. Suppose the $i-$th school sends B_i boys and G_i girls.

Let $B = \sum B_i$ and $G = \sum G_i$. We are given that $|B-G| = 1$.

The number of same sex matches is

$$\frac{1}{2}\sum B_i(B - B_i) + \frac{1}{2}\sum G_i(G - G_i) = \frac{1}{2}\left(B^2 - \sum B_i^2 + G^2 - \sum G_i^2\right)$$

The number of opposite sex matches is

$$\sum B_i(G - G_i) = BG - \sum B_i G_i$$

Thus we are given that

$$B^2 - \sum B_i^2 + G^2 - \sum G_i^2 - 2BG + 2\sum B_i G_i = 0 \text{ or } \pm 2.$$

Hence

$$(B - G)^2 - \sum (B_i - G_i)^2 = 0 \text{ or } \pm 2.$$

But $(B-G)^2 = 1$, so $\sum(B_i - G_i)^2 = \pm 1$ or 3. Since it cannot be negative, it must be 1 or 3. Hence $B_i = G_i$ except for 1 or 3 values of i, where $|B_i - G_i| = 1$.

Thus the largest number of schools that can have $B_i + G_i$ odd is 3.

3. Find the maximum possible value for the sum of the absolute values of the differences between each pair of n non-negative real numbers which do not exceed 1.

Solution: Let the numbers be $x_1, x_2, \ldots x_n$. Without loss of generality, we can assume that $x_1 \geq x_2 \cdots \geq x_n$. Then the sum S of absolute values of the differences between each pair

is

$$\begin{aligned}(x_1 - x_2) + (x_1 - x_3) + \cdots + (x_1 - x_n) \\ + (x_2 - x_3) + \cdots + (x_2 - x_n) \\ + (x_3 - x_4) + \cdots + (x_3 - x_n) \\ + \cdots + (x_{n-1} - x_n) \\ = (n-1)x_1 + (n-2)x_2 + (n-3)x_3 + \cdots + x_{n-1} \\ - (n-1)x_n - (n-2)x_{n-1} - \cdots - x_2 \\ = (n-1)x_1 + (n-3)x_2 + (n-5)x_3 + \cdots + (n-(2n-1))x_n\end{aligned}$$

Since all x_i are non-negative we maximize the sum by taking the x_i with non-positive coefficients to be zero and the x_i with positive coefficients to be 1.

That gives the maximum $S = (n-1)+(n-3)+(n-5)+\cdots$. If $n = 2m$, then $S = 1+3+5+\cdots+2m-1 = m^2$. If $n = 2m+1$, then $S = 2+4+\cdots+2m = m(m+1)$. We can combine that into a single formula as $S = \left[\frac{n^2}{4}\right]$.

4. If S is a sequence of positive integers let $p(S)$ be the product of the members of S. Let $m(S)$ be the arithmetic mean of $p(T)$ for all non-empty subsets T of S. S' is formed from S by appending an additional positive integer. If $m(S) = 13$ and $m(S') = 49$, find S'.

Solution: Suppose there are n elements in S, namely $a_1, a_2, \cdots, a_n$. Then

$$m(S) = \frac{K-1}{2^n - 1}$$

where $K = (a_1 + 1)(a_2 + 1)\cdots(a_n + 1)$.

If the additional integer added to S is h, then

$$m(S') = \frac{K(h+1) - 1}{2^{n+1} - 1}.$$

So we have

$$\begin{aligned} K &= 13(2^n - 1) + 1 \\ (h+1)K &= 49(2^{n+1} - 1) + 1 \end{aligned}$$

Hence

$$h + 1 = \frac{49 \cdot 2^{n+1} - 48}{13 \cdot 2^n - 12} = 8 - \frac{6 \cdot 2^n - 48}{13 \cdot 2^n - 12}.$$

and

$$\frac{6 \cdot 2^n - 48}{13 \cdot 2^n - 12}$$

must be an integer. For $n > 3$, $6 \cdot 2^n - 48 > 0$ and

$$13 \cdot 2^n - 12 = (6 \cdot 2^n - 48) + (7 \cdot 2^n + 36) > (6 \cdot 2^n - 48)$$

Hence for $n > 3$,

$$0 < \frac{6 \cdot 2^n - 48}{13 \cdot 2^n - 12} < 1$$

and consequently,

$$\frac{6 \cdot 2^n - 48}{13 \cdot 2^n - 12}$$

can not be an integer. Thus $n \leq 3$.

For $n = 1$,

$$\frac{6 \cdot 2^n - 48}{13 \cdot 2^n - 12} = -\frac{36}{14}$$

and for $n = 2$,

$$\frac{6 \cdot 2^n - 48}{13 \cdot 2^n - 12} = -\frac{24}{40}$$

If $n = 3$,

$$\frac{6 \cdot 2^n - 48}{13 \cdot 2^n - 12} = 0$$

and $h + 1 = 8$. Thus, $n = 3$ is the only possibility.

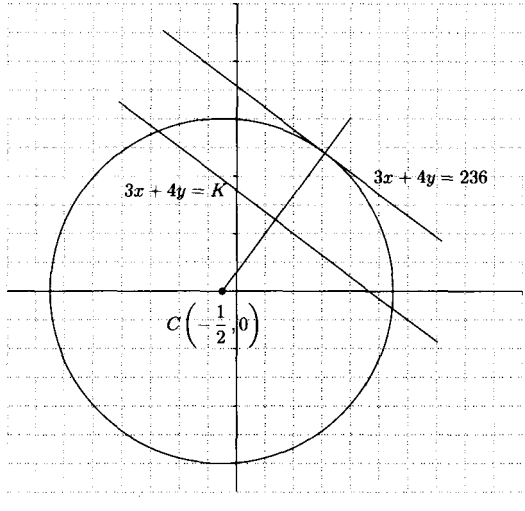
When $n = 3$, and the elements of S are a, b, c, we must have $(a+1)(b+1)(c+1) = 92 = 2 \cdot 2 \cdot 23$.

So $S = \{1, 1, 22\}$ and $S' = \{1, 1, 7, 22\}$.

5. Suppose the sum of first m consecutive positive even integers and n consecutive positive odd integers is 2256. Find the maximum value of $3m + 4n$.

Solution 1 The sum of first m consecutive positive even integers is $2 + 4 + \cdots + 2m = m(m+1)$ and the sum of first n consecutive positive odd integers is $1 + 3 + \cdots + (2n-1) = n^2$. Thus we are given that $m(m+1) + n^2 = 2256$ and we need to find the maximum value of $3m + 4n$.

Consider the circle $X^2 + Y^2 + X = 2256$. The problem can be



restated as follows:

Of all the lattice points (m, n) on this circle, for which point is the value of $3m + 4n$ maximum?

Clearly, the circle has center at $(-\frac{1}{2}, 0)$ and radius $\sqrt{\frac{1}{4} + 2256} = \frac{95}{2}$. We need to find the farthest point at which the line of the form $3X + 4Y = K$ meets this circle. If there is a unique such point, then necessarily the line $3X + 4Y = K$ will

be a tangent to this circle and hence we can find such a point by looking at the point at which a perpendicular to the line from the center of the circle meets the circle. The line through the center perpendicular to $3X + 4Y = K$ has the equation

$$4\left(X + \frac{1}{2}\right) - 3Y = 0$$

This line meets the circle at a point (X, Y) given by

$$\left(X + \frac{1}{2}\right)^2 + \frac{16}{9}\left(X + \frac{1}{2}\right)^2 = \left(\frac{95}{2}\right)^2$$

Hence we get $X + \frac{1}{2} = \frac{57}{2}$ and corresponding $Y-$coordinate is $Y = \frac{4}{3}\left(X + \frac{1}{2}\right) = \frac{4}{3} \times \frac{57}{2} = 38$. Hence the point $(28, 38)$ lies on the circle and $3X + 4Y = 236$ is a tangent to the circle at that point. Thus $3m + 4n$ is maximum when $m = 28$ and $n = 38$ and the maximum value is 236.

Clearly, $(2 + 4 + \cdots + 56) + (1 + 3 + \cdots + 75) = 2256$ verifying the condition given in the problem.

Solution 2 We use a heavy artillery gun – Lagrange multipliers.

This is a problem of finding constrained maximum. We need to find the maximum of $3x+4y$ under the constraint $x^2+y^2+x = 2256$. Thus we consider the function

$$f(x, y, \lambda) = 3x + 4y - \lambda(x^2 + y^2 + x - 2256)$$

where λ is the Lagrange multiplier.

Differentiating partially with respect to x, y, λ, we obtain

$$\begin{aligned}\frac{\partial f}{\partial x} &= 3-\lambda(2x+1)\\ \frac{\partial f}{\partial y} &= 4-\lambda(2y)\\ \frac{\partial f}{\partial \lambda} &= -(x^2+y^2+x-2256)\end{aligned}$$

Equating the above to zero, we get

$$\lambda = \frac{3}{2x+1} = \frac{4}{2y}$$

Hence $3y = 4\left(x+\frac{1}{2}\right)$. Substituting in $x^2+y^2+x-2256=0$, we get

$$\left(x+\frac{1}{2}\right)^2 + \frac{16}{9}\left(x+\frac{1}{2}\right)^2 = 2256+\frac{1}{4} = \left(\frac{95}{2}\right)^2$$

The rest of the proof is as in **Solution 1**.

6. Find

$$\frac{1+\frac{1}{2^p}+\frac{1}{3^p}+\frac{1}{4^p}+\cdots}{1-\frac{1}{2^p}+\frac{1}{3^p}-\frac{1}{4^p}+\cdots} \quad \text{and} \quad \lim_{p\to\infty}\frac{1+\frac{1}{2^p}+\frac{1}{3^p}+\frac{1}{4^p}+\cdots}{1-\frac{1}{2^p}+\frac{1}{3^p}-\frac{1}{4^p}+\cdots}.$$

Solution: Let

$$\begin{aligned}A &= 1+\frac{1}{3^p}+\frac{1}{5^p}+\cdots\\ B &= \frac{1}{2^p}+\frac{1}{4^p}+\frac{1}{6^p}+\cdots\end{aligned}$$

Then $2^pB = A+B$ and so $A=(2^p-1)B$. Therefore

$$\frac{1+\frac{1}{2^p}+\frac{1}{3^p}+\frac{1}{4^p}+\cdots}{1-\frac{1}{2^p}+\frac{1}{3^p}-\frac{1}{4^p}+\cdots} = \frac{A+B}{A-B} = \frac{2^pB}{(2^p-2)B} = \frac{2^p}{2^p-2}.$$

Thus,

$$\lim_{p\to\infty}\frac{1+\frac{1}{2^p}+\frac{1}{3^p}+\frac{1}{4^p}+\cdots}{1-\frac{1}{2^p}+\frac{1}{3^p}-\frac{1}{4^p}+\cdots} = \lim_{p\to\infty}\frac{2^p}{(2^p-2)} = \lim_{p\to\infty}\frac{1}{(1-\frac{1}{2^{p-1}})} = 1.$$

REGIONAL MATHEMATICAL OLYMPIAD – 2013

1. Let ABC be an acute angled triangle. The circle Γ with BC as diameter intersects AB and AC again at P and Q, respectively. Determine $\angle BAC$ given that the orthocenter of triangle APQ lies on Γ.

   Solution: Let K denote the orthocenter of triangle APQ. Since triangles ABC and AQP are similar it follows that K lies in the interior of triangle APQ.

   Note that $\angle KPA = \angle KQA = 90^\circ - \angle A$. Since $BPKQ$ is a cyclic quadrilateral it follows that $\angle BQK = 180^\circ - \angle BPK = 90^\circ - \angle A$, while on the other hand $\angle BQK = \angle BQA - \angle KQA = \angle A$ since BQ is perpendicular to AC. This shows that $90^\circ - \angle A = \angle A$, so $\angle A = 45^\circ$.

2. Let $f(x) = x^3+ax^2+bx+c$ and $g(x) = x^3+bx^2+cx+a$, where a, b, c are integers with $c \neq 0$. Suppose that the following conditions hold:

   (a) $f(1) = 0$;

   (b) the roots of $g(x)$ are squares of the roots of $f(x)$.

   Find the value of $a^{2013} + b^{2013} + c^{2013}$.

   Solution: Note that $g(1) = f(1) = 0$, so 1 is a root of both $f(x)$ and $g(x)$. Let p and q be the other two roots of $f(x)$, so p^2 and q^2 are the other two roots of $g(x)$. We then get $pq = -c$ and $p^2q^2 = -a$, so $a = -c^2$. Also, $(-a)^2 = (p+q+1)^2 = p^2+q^2+1+2(pq+p+q) = -b+2b = b$. Therefore $b = c^4$. Since $f(1) = 0$ we therefore get $1+c-c^2+c^4 = 0$. Factorising, we get $(c+1)(c^3 - c^2 + 1) = 0$. Note that $c^3 - c^2 + 1 = 0$ has no integer root and hence $c = -1; b = 1; a = -1$. Therefore $a^{2013} + b^{2013} + c^{2013} = -1$.

3. Find all primes p and q such that p divides $q^2 - 4$ and q divides $p^2 - 1$.

   Solution: Suppose that $p \leq q$. Since q divides $(p-1)(p+1)$ and $q > p - 1$ it follows that q divides $p + 1$ and hence $q = p + 1$. Therefore $p = 2$ and $q = 3$.

   On the other hand, if $p > q$ then p divides $(q - 2)(q + 2)$ implies that p divides $q + 2$ or $q - 2 = 0$. This gives either $p = q + 2$ or $q = 2$. In the former case it follows that that q divides $(q + 2)^2 - 1$, so q divides 3. This gives the solutions $p > 2, q = 2$ and $(p, q) = (5, 3)$.

4. Find the number of 10-tuples $(a_1, a_2, \cdots, a_{10})$ of integers such that $|a_1| \leq 1$ and

   $$a_1^2+a_2^2+a_3^2+\cdots+a_{10}^2-a_1a_2-a_2a_3-a_3a_4-\cdots-a_9a_{10}-a_{10}a_1 = 2.$$

   Solution: Let $a_{11} = a_1$. Multiplying the given equation by 2 we get

   $$(a_1 - a_2)^2 + (a_2 - a_3)^2 + \cdots + (a_{10} - a_1)^2 = 4$$

   Note that if $a_i - a_i + 1 = \pm 2$ for some $i = 1, \cdots, 10$, then $a_j - a_{j+1} = 0$ for all $j \neq i$ which contradicts the equality $\sum_{i=1}^{10}(a_i - a_{i+1}) = 0$. Therefore $a_i - a_{i+1} = 1$ for exactly two values of i in $\{1, 2, \cdots, 10\}, a_i - a_{i+1} = -1$ for two other values of i and $a_i - a_{i+1} = 0$ for all other values of i. There are $\binom{10}{2} \times \binom{8}{2} = 45 \times 28$ possible ways of choosing these values. Note that $a_1 = -1, 0$ or 1, so in total there are $3 \times 45 \times 28$ possible integer solutions to the given equation.

5. Let ABC be a triangle with $\angle A = 90°$ and $AB = AC$. Let D and E be points on the segment BC such that $BD : DE : EC = 3 : 5 : 4$. Prove that $\angle DAE = 45°$.

   Solution: Rotating the configuration about A by 90°, the point B goes to the point C. Let P denote the image of

the point D under this rotation. Then $CP = BD$ and $\angle ACP = \angle ABC = 45°$, so ECP is a right-angled triangle with $CE : CP = 4 : 3$. Hence $PE = ED$. It follows that $ADEP$ is a kite with $AP = AD$ and $PE = ED$. Therefore AE is the angular bisector of $\angle PAD$. This implies that $\angle DAE = \angle PAD/2 = 45°$.

6. Suppose that m and n are integers such that both the quadratic equations $x^2 + mx - n = 0$ and $x^2 - mx + n = 0$ have integer roots. Prove that n is divisible by 6.

Solution: Let a be an integer. If a is not divisible by 3 then $a^2 \equiv 1 \pmod 3$, i.e., 3 divides $a^2 - 1$, and if a is odd then $a^2 \equiv 1 \pmod 8$, i.e., 8 divides $a^2 - 1$.

Note that the discriminants of the two quadratic polynomials are both squares of integers. Let a and b be integers such that $m^2 - 4n = a^2$ and $m^2 + 4n = b^2$. Therefore $8n = b^2 - a^2$ and $2m^2 = a^2 + b^2$. If 3 divides m then 3 divides both a and b, so 3 divides n. On the other hand if 3 does not divide m then 3 does not divide a or b. Therefore 3 divides $b^2 - a^2$ and hence 3 divides n.

If m is odd, then so is a, and therefore $4n = m^2 - a^2$ is divisible by 8, so n is even. On the other hand, if m is even then both a and b are even. Further $(m/2)^2 - n = (a/2)^2$ and $(m/2)^2 + n = (b/2)^2$, so $(b - a)/2$ is even. In particular, $n = (b^2 - a^2)/4$ is even.

29th INDIAN NATIONAL MATHEMATICAL OLYMPIAD – 2014

1. In a triangle ABC, let D be a point on the segment BC such that $AB + BD = AC + CD$. Suppose that the points B, C and the centroids of triangles ABD and ACD lie on a circle. Prove that $AB = AC$.

   Solution: Let G_1, G_2 denote the centroids of triangles ABD and ACD. Then G_1, G_2 lie on the line parallel to BC that passes through the centroid of triangle ABC. Therefore BG_1G_2C is an isosceles trapezoid. Therefore it follows that $BG_1 = CG_2$. This proves that $AB^2 + BD^2 = AC^2 + CD^2$. Hence it follows that $AB \cdot BD = AC \cdot CD$. Therefore the sets $\{AB, BD\}$ and $\{AC, CD\}$ are the same (since they are both equal to the set of roots of the same polynomial). Note that if $AB = CD$ then $AC = BD$ and then $AB + AC = BC$, a contradiction. Therefore it follows that $AB = AC$.

2. Let n be a natural number. Prove that

$$\left[\frac{n}{1}\right] + \left[\frac{n}{2}\right] + \left[\frac{n}{3}\right] + \cdots \left[\frac{n}{n}\right] + \left[\sqrt{n}\right]$$

   is even. (Here $[x]$ denotes the largest integer smaller than or equal to x.)

   Solution: Let $f(n)$ denote the given expression. Then $f(1) = 2$ which is even. Now suppose that $f(n)$ is even for some $n \geq 1$. Then

$$f(n+1) = \left[\frac{n+1}{1}\right] + \left[\frac{n+1}{2}\right] + \left[\frac{n+1}{3}\right] + \cdots \left[\frac{n+1}{n+1}\right] + \left[\sqrt{n+1}\right]$$

$$= \left[\frac{n}{1}\right] + \left[\frac{n}{2}\right] + \left[\frac{n}{3}\right] + \cdots \left[\frac{n}{n}\right] + \left[\sqrt{n+1}\right] + \sigma(n+1),$$

   where $= \sigma(n+1)$ denotes the number of positive divisors of $n+1$. This follows from $\left[\frac{n+1}{1}\right] = \left[\frac{n}{k}\right] + 1$ if k divides $n+1$, and

$\left[\frac{n+1}{k}\right] = \left[\frac{n}{k}\right]$ otherwise. Note that $\left[\sqrt{n+1}\right] = [\sqrt{n}]$ unless $n+1$ is a square, in which case $\left[\sqrt{n+1}\right] = [\sqrt{n}] + 1$. On the other hand $\sigma(n+1)$ is odd if and only if $n+1$ is a square. Therefore it follows that $f(n+1) = f(n)+2l$ for some integer l. This proves that $f(n+1)$ is even. Thus it follows by induction that $f(n)$ is even for all natural number n.

3. Let a, b be natural numbers with $ab > 2$. Suppose that the sum of their greatest common divisor and least common multiple is divisible by $a+b$. Prove that the quotient is at most $(a+b)/4$. When is this quotient exactly equal to $(a+b)/4$?

   Solution: Let g and l denote the greatest common divisor and the least common multiple, respectively, of a and b. Then $gl = ab$. Therefore $g+l \leq ab+1$. Suppose that $(g+l)/(a+b) > (a+b)/4$. Then we have $ab+1 > (a+b)^2/4$, so we get $(a-b)^2 < 4$. Assuming, $a \geq b$ we either have $a = b$ or $a = b+1$. In the former case, $g = l = a$ and the quotient is $(g+l)/(a+b) = 1 \leq (a+b)/4$. In the latter case, $g = 1$ and $l = b(b+1)$ so we get that $2b+1$ divides b^2+b+1. Therefore $2b+1$ divides $4(b^2+b+1) - (2b+1)^2 = 3$ which implies that $b = 1$ and $a = 2$, a contradiction to the given assumption that $ab > 2$. This shows that $(g+l)/(a+b) \leq (a+b)/4$. Note that for the equality to hold, we need that either $a = b = 2$ or, $(a-b)^2 = 4$ and $g = 1, l = ab$. The latter case happens if and only if a and b are two consecutive odd numbers. (If $a = 2k+1$ and $b = 2k-1$ then $a+b = 4k$ divides $ab+1 = 4k^2$ and the quotient is precisely $(a+b)/4$.)

4. Written on a blackboard is the polynomial $x^2+x+2014$. Calvin and Hobbes take turns alternatively (starting with Calvin) in the following game. During his turn, Calvin should either increase or decrease the coefficient of x by 1. And during his turn, Hobbes should either increase or decrease the constant term by 1. Calvin wins if at any point of time the polynomial

on the blackboard at that instant has integer roots. Prove that Calvin has a winning strategy.

Solution: For $i \geq 0$, let $f_i(x)$ denote the polynomial on the blackboard after Hobbes' i-th turn. We let Calvin decrease the coefficient of x by 1. Therefore $f_{i+1}(2) = f_i(2) - 1$ or $f_{i+1}(2) = f_i(2) - 3$ (depending on whether Hobbes increases or decreases the constant term). So for some i, we have $0 \leq f_i(2) \leq 2$. If $f_i(2) = 0$ then Calvin has won the game. If $f_i(2) = 2$ then Calvin wins the game by reducing the coefficient of x by 1. If $f_i(2) = 1$ then $f_{i+1}(2) = 0$ or $f_{i+1}(2) = -2$. In the former case, Calvin has won the game and in the latter case Calvin wins the game by increasing the coefficient of x by 1.

5. In an acute-angled triangle ABC, a point D lies on the segment BC. Let O_1, O_2 denote the circumcentres of triangles ABD and ACD, respectively. Prove that the line joining the circumcentre of triangle ABC and the orthocentre of triangle O_1O_2D is parallel to BC.

Solution: Without loss of generality assume that $\angle ADC \geq 90°$. Let O denote the circumcenter of triangle ABC and K the orthocentre of triangle O_1O_2D. We shall first show that the points O and K lie on the circumcircle of triangle AO_1O_2. Note that circumcircles of triangles ABD and ACD pass through the points A and D, so AD is perpendicular to O_1O_2 and, triangle AO_1O_2 is congruent to triangle DO_1O_2. In particular, $\angle AO_1O_2 = \angle O_2O_1D = \angle B$ since O_2O_1 is the perpendicular bisector of AD. On the other hand since OO_2 is the perpendicular bisector of AC it follows that $\angle AOO_2 = \angle B$. This shows that O lies on the circumcircle of triangle AO_1O_2. Note also that, since AD is perpendicular to O_1O_2, we have $\angle O_2KA = 90° - \angle O_1O_2K = \angle O_2O_1D = \angle B$. This proves that K also lies on the circumcircle of triangle AO_1O_2. Therefore $\angle AKO = 180° - \angle AO_2O = \angle ADC$ and hence OK

is parallel to BC.

Remark. The result is true even for an obtuse-angled triangle.

6. Let n be a natural number and $X = \{1, 2, \cdots, n\}$. For subsets A and B of X we define $A \triangle B$ to be the set of all those elements of X which belong to exactly one of A and B. Let $\mathcal{F}$ be a collection of subsets of X such that for any two distinct elements A and B in $\mathcal{F}$ the set $A \triangle B$ has at least two elements. Show that $\mathcal{F}$ has at most 2^{n-1} elements. Find all such collections $\mathcal{F}$ with 2^{n-1} elements.

Solution: For each subset A of $\{1, 2, \cdots, n-1\}$, we pair it with $A \cup \{n\}$. Note that for any such pair (A, B) not both A and B can be in $\mathcal{F}$. Since there are 2^{n-1} such pairs it follows that $\mathcal{F}$ can have at most 2^{n-1} elements. We shall show by induction on n that if $|\mathcal{F}| = 2^{n-1}$ then $\mathcal{F}$ contains *either all the subsets with odd number of elements or all the subsets with even number of elements*. The result is easy to see for $n = 1$. Suppose that the result is true for $n = m - 1$. We now consider the case $n = m$. Let $\mathcal{F}_1$ be the set of those elements in $\mathcal{F}$ which contain m and $\mathcal{F}_2$ be the set of those elements which do not contain m. By induction, $\mathcal{F}_2$ can have at most 2^{m-2} elements. Further, for each element A of $\mathcal{F}_1$ we consider $A \backslash \{n, m\}$. This new collection also satisfies the required property, so it follows that $\mathcal{F}_1$ has at most 2^{m-2} elements. Thus, if $|\mathcal{F}| = 2^{m-1}$ then it follows that $|\mathcal{F}_1| = |\mathcal{F}_2| = 2^{m-2}$. Further, by induction hypothesis, $\mathcal{F}_2$ contains all those subsets of $\{1, 2, \cdots, m-1\}$ with (say) even number of elements. It then follows that $\mathcal{F}_1$ contains all those subsets of $\{1, 2, \cdots, m\}$ which contain the element m and which contains an even number of elements. This proves that $\mathcal{F}$ contains either all the subsets with odd number of elements or all the subsets by even number of elements.

Report on Association Activities

1. The association conducted four workshops in 2014. All the workshops were conducted in our new building.

Audience	From	To	Number of participants
Teachers	7 May 2014	10 May 2014	26
Primary	14 May 2014	17 May 2014	33
Sub junior	21 May 2014	24 May 2014	44
RMO	30 April 2014	3 May 2014	57

Boarding and lodging facilities for outstation participants were provided. On the last day of the workshops for students, a test was conducted and the top three scorers were awarded prizes.

2. This year marks the 50th year of our association. We plan to celebrate by conducting 50 workshops. We have started compiling the details of the workshops in which our resource persons have participated. As on date (17 June 2014), we have already participated in more than ten workshops. There are proposals from some organizations offering to host some of the workshops. We will provide details in our official website.

3. 49th conference is scheduled in Bhopal during 27th to 29th December 2014. Formal call notice will be issued by September 1st with 1st October as last date for registrations. Details will also be available in the website amtionline.com from 1st September. Please register early.

4. Plans are underway to modify and increase the content of our website. Your continued patronage is solicited.

5. Five new solution books have been added to our publications. They are

1) Gems Primary-II
2) Gems Sub- Junior-II

3) Gems Junior -II
4) Gems Inter II
5) RMO-INMO

These contain problems and solutions from the competitions conducted by AMTI from 2004 onwards. Each book is priced at Rs. 150/-.

The next few pages show some photographs taken during the workshops conducted in Chennai and at Bajaj Science Centre, Wardha.

General Secretary

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